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Integer-forcing in multiterminal coding: uplink-downlink duality and source-channel duality

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INTEGER-FORCING IN MULTITERMINAL CODING: UPLINK-DOWNLINK DUALITY AND SOURCE-CHANNEL DUALITY

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INTEGER-FORCING IN MULTITERMINAL CODING: UPLINK-DOWNLINK DUALITY AND SOURCE-CHANNEL DUALITY

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ABSTRACT

Interference is considered to be a major obstacle to wireless communication. Popular approaches, such as the zero-forcing receiver in MIMO (multiple-input and multiple-output) multiple-access channel (MAC) and zero-forcing (ZF) beamforming in MIMO broadcast channel (BC), eliminate the interference first and decode each codeword separately using a conventional single-user decoder. Recently, a transceiver architecture called integer-forcing (IF) has been proposed in the context of the MIMO Gaussian multiple-access channel to exploit integer-linear combinations of the codewords. Instead of treating other codewords as interference, the integer-forcing approach decodes linear combinations of the codewords from different users and solves for desired codewords. Integer-forcing can closely approach the performance of the optimal joint maximum likelihood decoder. An advanced version called successive integer-forcing can achieve the sum capacity of the MIMO MAC channel. Several extensions of integer-forcing have been developed in various scenarios, such as integer-forcing for the Gaussian MIMO broadcast channel, integer-forcing for Gaussian distributed source coding and integer-forcing interference alignment for the Gaussian interference channel.

This dissertation demonstrates duality relationships for integer-forcing among three different channel models. We explore in detail two distinct duality types in this thesis: uplink-downlink duality and source-channel duality. Uplink-downlink duality is established for integer-forcing between the Gaussian MIMO multiple-access channel and its dual Gaussian MIMO broadcast channel. We show that under a total power constraint, integer-forcing can achieve the same sum rate in both cases. We further develop a dirty-paper integer-forcing scheme for the Gaussian MIMO BC and show an uplink-downlink duality with successive integer-forcing for the Gaussian MIMO MAC. The source-channel duality is established for integer-forcing between the Gaussian MIMO multiple-access channel and its dual Gaussian distributed source coding problem. We extend previous results for integer-forcing source coding to allow for successive cancellation. For integer-forcing without successive cancellation in both channel coding and source coding, we show the rates in two scenarios lie within a constant gap of one another. We further show that there exists a successive cancellation scheme such that both integer-forcing channel coding and integer-forcing source coding achieve the same rate tuple.

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List of Abbreviations

AWGN	....	Additive white Gaussian noise
BC	....	Broadcast channel
CSI	....	Channel state information
CSIR	....	Channel state information at receiver
CSIT	....	Channel state information at transmitter
DoF	....	Degrees of freedom
DPC	....	Dirty-paper coding
DPC-IF	....	Dirty-paper integer-forcing
IF	....	Integer-forcing
IFIA	....	Integer-forcing interference alignment
LLL algorithm	....	Lenstra-Lenstra-Lovász algorithm
MAC	....	Multiple-access channel
MIMO	....	Multiple-input and multiple-output
MMSE	....	Minimum mean square error
SIC	....	Successive interference cancellation
SIC-IF	....	Successive integer-forcing (channel coding)
SIC-IFSC	....	Successive integer-forcing source coding
SINR	....	Signal-to-interference-plus-noise ratio
SISO	....	Single-input and single-output
SNR	....	Signal-to-noise ratio
ZF	....	Zero-forcing

Chapter 1

Introduction

1.1 Motivation

Interference is one of the major issues in wireless networks when multiple communications are conducted over a common medium. Methods to manage interference have been proposed for various models, such as the multiple-access channel (MAC), broadcast channel (BC), relay networks, interference channel and cellular networks. In cellular networks, base stations separate desired messages from interference by performing transmissions on different frequency bands. In satellite systems, signals are divided into different time slots in order to avoid interference from each other. In traditional designs for multiterminal systems, interference is eliminated by treating interference as noise, e.g., linear equalization is used in the MAC to maximize the SINR (signal-to-interference-plus-noise ratio) and linear beamforming is applied in the BC to align interference for the same purpose.

The complexity of the transmitter/receiver is another issue in wireless communication design. Many capacity-approaching schemes, such as i.i.d. random coding and joint typicality decoding, require an exhaustive search through the codebook, meaning that the complexity increases exponentially with the codeword's blocklength. It is of considerable interest to find low complexity schemes that operate at or near capacity. To eliminate interference with low computational complexity, zero-forcing (ZF) is used as a MIMO MAC equalization strategy and MIMO BC beamforming strategy. By inverting the channel matrix at the receiver side for MAC or pre-inverting the channel at the transmitter side for BC, ZF can create a point-to-point interference-free channel for each data stream. As a result, each data stream can be encoded and decoded using a single-input single-output (SISO) algorithm. Although channel inversion can completely eliminate interference, it often does so at the cost of significantly reducing the SNR for the desired codeword.

Rather than eliminating interference before decoding, an alternative approach is to decode linear combinations of interfering codewords. The recent development of compute-and-forward [Nazer and Gastpar, 2011] and integer-forcing (IF) [Zhan et al., 2014] provide a path for this option. Consider a MIMO MAC where all data streams are generated using the same structured codebook, such as a lattice codebook. Note that, for a lattice codebook, an integer linear combination of codewords is still a valid codeword. The receiver observes the interfering data streams and attempts to decode the integer-linear combination that most closely approximates the real linear combination taken by the channel. With a sufficient number of linearly independent combinations decoded, the receiver can solve for each message. The effective noise is determined by how well the integer matrix approximates the channel matrix.

Conventional approaches, like ZF, eliminate interference by creating an effective identity channel matrix through equalization and beamforming. On the other hand, IF only requires transforming the channel into any full rank integer matrix $\mathbf{A}$ such that messages can be recovered from the combinations. By choosing the integer matrix $\mathbf{A}$ properly, IF can increase the SINR at the receiver side in both MAC and BC. With lower effective noise, IF outperforms ZF. Since for structured coding a combination of codewords is still a valid codeword, the decoding/encoding complexity for IF is equivalent to the decoding/encoding complexity for a SISO channel, i.e., the complexity of IF is close to that of ZF.

IF can be applied in multiple channel models and coding schemes. Specifically, IF coding schemes have been developed for the Gaussian MIMO MAC [Zhan et al., 2014, Ordentlich et al., 2013,Ordentlich et al., 2012,Ntranos et al., 2013a], Gaussian MIMO BC [Hong and Caire, 2012] and distributed source coding problem [Ordentlich and Erez, 2013]. These coding schemes, although designed for different models, show some similarities in channel equalization, beamforming and lattice code design. Among these models, a problem might be well explored in one of the models but remain unsolved for the rest. For example, the equalization problem for IF has a closed form optimal solution in the MAC but remains as a non-convex optimization problem in the BC, and the IF beamforming problem is solved in closed form in the BC but remains non-convex in the MAC. If connections can be established among different IF models, the existing results in one model can be delivered to another model as solutions for unsolved problems or as inspirations for coding scheme design. Such a connection between different coding schemes and system models is called duality.

1.2 State of the art

1.2.1 Integer-forcing (IF) in multiterminal coding problems

The family line of integer-forcing can be traced back to lattice codes and their applications in communications. For many additive white Gaussian noise (AWGN) channels, nested lattice codes can approach the performance of standard random coding strategies. For instance, nested lattice codes achieve the capacity of the AWGN point-to-point channel [Erez and Zamir, 2004] and can achieve the diversity-multiplexing trade-off of MIMO channels [Gamal et al., 2004]. One advantage of a lattice code is that its algebraic structure ensures that an integer combination of lattice codewords is itself a valid lattice codeword. Based on lattice coding, compute-and-forward was introduced in [Nazer and Gastpar, 2011] as a relaying strategy for distributed MIMO relay networks. In compute-and-forward, all messages are encoded using the same lattice codebook and each relay decodes a linear combination of interfering messages. Relays then forward combinations to a central base station where the combinations are solved as linear equations for the desired messages. Compared to classical relaying strategies like decode-and-forward [Cover and Gamal, 1979, Laneman et al., 2004, Kramer et al., 2005, Gamal et al., 2007], compress-and-forward [Cover and Gamal, 1979, Kramer et al., 2005, Kim, 2008, Aleksic et al., 2009, Sanderovich et al., 2009, Lim et al., 2011] and amplify-and-forward [Laneman et al., 2004, Gamal et al., 2007, Gastpar and Vetterli, 2005, Borade et al., 2007], compute-and-forward shows advantages in moderate signal-to-noise ratio (SNR) regimes. The performance of compute-and-forward increases when the channel matrix is close to an integer matrix. To create an integer channel matrix, integer-forcing [Zhan et al., 2014] was introduced. By allowing the receiver to equalize the channel to any full-rank integer matrix instead of the identity matrix, integer-forcing linear receiver outperforms traditional linear receivers like the minimum mean square error (MMSE) receiver and ZF linear receiver.

Following [Zhan et al., 2014], several extensions of integer-forcing have been developed. For the K-user Gaussian MIMO MAC, integer-forcing can approach the sum capacity to within a constant gap of $\frac{K}{2}\log K$ [Ordentlich et al., 2012]. The results can be generalized for unequal (or asymmetric) power allocation across transmitters [Ntranos et al., 2013a]. An advanced technique named successive integer-forcing [Ordentlich et al., 2013] was proposed to achieve the sum capacity of the Gaussian MIMO MAC. Integer-forcing coupled with space-time codes was proposed in [Ordentlich and Erez, 2015a], which universally approaches the capacity of any Gaussian MIMO channel up to a constant gap that depends only on the number of transmit antennas. The above achievability and optimality results are unified and generalized in [Nazer et al., 2016] and we point interested readers to that paper for more details.

Extending integer-forcing into other communication models emerges as an interesting direction. Three models recently attract particular attention: the Gaussian MIMO BC (or downlink channel), the Gaussian distributed source coding problem and the Gaussian interference channel. For the downlink MIMO BC channel, [Hong and Caire, 2012] showed that integer-forcing can be employed as a beamforming strategy. For the Gaussian distributed source coding problem, [Ordentlich and Erez, 2013] demonstrated how integer-forcing can be used as a lattice-based distributed equalization scheme. Integer-forcing can also be used as an interference alignment technique [Ntranos et al., 2013b] for the Gaussian MIMO interference channel. Specifically, integer-forcing interference alignment (IFIA) can exploit both signal-space alignment and signal-scale alignment while conventional alignment schemes only exploit either one or the other. In this dissertation, we propose a class of iterative optimization algorithms for IFIA and show that IFIA outperforms existing algorithm like Max-SINR algorithm [Gomadam et al., 2011] when linear strategies are not feasible in a degrees-of-freedom sense [Yetis et al., 2010].

1.2.2 Duality in multiterminal coding problems

The history of duality in information theory dates back to Shannon's landmark paper on rate-distortion theory [Shannon, 1959]. Shannon pointed out the similarity between the data compression problem and data transmission problem which can be studied as information-theoretic duals between source coding and channel coding. Cover and Thomas formulated the source-channel duality in [Cover and Thomas, 2006, Page. 324-325] using an interpretation of packing versus covering. Later, source-channel duality was extended to multiterminal channels [Pradhan et al., 2002, Yu, 1998] and source coding with side information [Pradhan et al., 2003, Barron et al., 2003, Cover and Chiang, 2002].

For multiterminal channels, there is another form of duality between the BC and the MAC known as uplink-downlink duality. (Uplink corresponds to the MAC and downlink corresponds to the BC.) Uplink-downlink duality was introduced in [Jindal et al., 2003] and has been used to characterized the sum capacity of the vector Gaussian broadcast channel [Viswanath and Tse, 2003, Vishwanath et al., 2003, Yu and Cioffi, 2004]. It can be shown that the capacity of the Gaussian BC can be expressed by the capacity region of a reciprocal Gaussian MAC with only a total power constraint. Similar results are established for the deterministic BC and deterministic MAC in [Jindal et al., 2004]. A re-interpretation of uplink-downlink duality is given in [Yu, 2006] using Lagrangian duality in a minimax optimization approach.

Combining uplink-downlink duality and source-channel duality, a “duality loop” can be created for four multiterminal coding problems: broadcast channel coding, multiple-access channel coding, multiterminal source coding [Berger, 1977] and multiple description (MD) coding [Gamal and Cover, 1982]. We point interested readers to [Stanković et al., 2006] for more details. Duality builds connections between different problems. In many scenarios, duality can map algorithms, schemes or simply intuitions from one problem to another. When there are well-developed results in one model, duality can be used to either develop or inspire new solutions to the dual problem. Many motivations and inspirations in channel and source coding come from duality. To list but a few, the sum capacity of the vector Gaussian broadcast channel [Viswanath and Tse, 2003, Vishwanath et al., 2003, Yu and Cioffi, 2004] was characterized from the MAC sum capacity using uplink-downlink duality, trellis-coded quantization [Marcellin et al., 1990] was inspired from trellis-coded modulation [Ungerboeck, 1982] using source-channel duality and the Max-SINR algorithm [Gomadam et al., 2011] for Gaussian interference channel is based on the idea of duality in channel reciprocity.

There are three main types of dualities in multiterminal channels: *formula dual-*

ity, *functional duality* and *operational duality*. A good summary of these dualities is given in [Stanković et al., 2006]. *Formula duality* is established upon the expressions for achievable rates in two dual coding problems. It is also referred as random coding level duality in [Wang and Viswanath, 2003]. Source-channel duality in [Cover and Chiang, 2002, Yu, 1998] and uplink-downlink duality in [Viswanath and Tse, 2003, Vishwanath et al., 2003, Jindal et al., 2004] all fall into the category of *formula duality*. *Functional duality*, compare to *formula duality*, is a stronger connection which provides insights about the encoders and decoders. For *functional duality*, the optimal encoder/decoder in one problem can be identically mapped from the optimal decoder/encoder of another. Examples of *functional duality* are given in [Pradhan et al., 2003, Pradhan et al., 2002] between channel and source coding. *Operational duality* is a special duality developed for deterministic binning schemes. It is established between source and channel coding where one uses the same maximal codebook by reversing the operation of encoding and decoding from another [Wang and Viswanath, 2003].

This dissertation develops two dualities for integer-forcing: one source-channel duality and one uplink-downlink duality. Both dualities in this dissertation are *formula duality* since they are established according to the rate expressions of IF. However, both dualities also provide insight between different coding schemes in different communication models such as channel equalization, beamforming, integer combination selections and successive cancellation strategies, which lead to a stronger connection in both cases. Further development of *functional duality* for IF can be considered as an interesting direction for future works.

1.3 Contributions

The main contributions of this dissertation focus on three integer-forcing problems. Two of them are directly related to duality establishment: uplink-downlink duality for integer-forcing and source-channel duality for integer-forcing. Both dualities are developed as *formula duality* along with additional insights into connections of encoding scheme design and decoding scheme design. Fig. 1·1 shows the relationships among the two dualities as well as the channel models they are connected with. Fig. 1·1 also shows a potential "duality ring" establishment in future work.

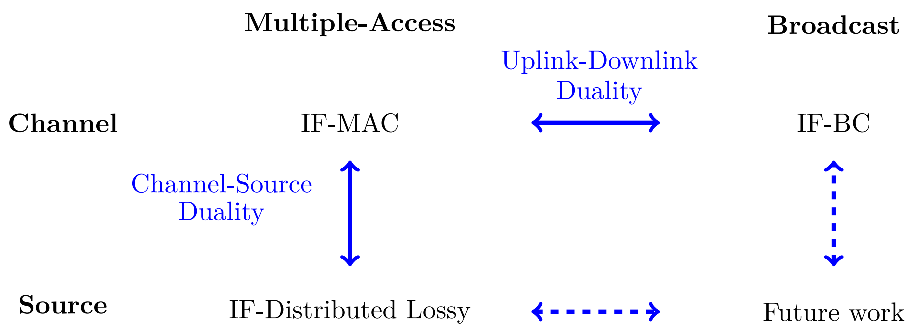

Figure 1·1: Overview of duality relationships for integer-forcing (IF)

Before the establishment of dualities, modifications and extensions for existing coding schemes are needed in both IF BC channel coding and IF source coding. Table 1.1 list the modifications and extensions proposed in this dissertation as well as corresponding results from previous works.

This dissertation also proposes a class of iterative optimization algorithms for integer-forcing interference alignment (IFIA) inspired by the idea of duality. We provide details about our contributions in the rest of this section and summarize them for each problem separately.

	IF-MAC	IF-BC	IF Source Coding
Symmetric rates, equal powers	[Nazer and Gastpar, 2011, Zhan et al., 2014]	[Hong and Caire, 2012]	[Ordentlich and Erez, 2013]
Asymmetric rates, equal powers	[Ordentlich et al., 2014]	This dissertation	This dissertation
Asymmetric rates, equal powers, successive cancellation	[Ordentlich et al., 2013]	This dissertation	This dissertation
Asymmetric rates, unequal powers, successive cancellation	[Nazer et al., 2016]	This dissertation	This dissertation

Table 1.1: Modifications and extensions for IF coding schemes

1.3.1 Uplink-downlink duality for IF

IF in MAC is well studied in [Zhan et al., 2014, Ordentlich et al., 2012, Ordentlich et al., 2013, Nazer et al., 2016]. Specifically, successive integer-forcing (SIC-IF) [Ordentlich et al., 2013] can achieve the sum capacity of Gaussian MIMO MAC. On the other hand, only limited work has been done for IF in BC [Hong and Caire, 2012]. The establishment of IF uplink-downlink duality is performed in a two-stage development: the modification of reverse compute-and-forward in [Hong and Caire, 2012] and uplink-downlink duality establishment. Finally, several by-products are generated from the duality results.

- Modification of reverse compute-and-forward:
  - We expand works in [Hong and Caire, 2012] to unequal powers.
  - We expand works in [Hong and Caire, 2012] to asymmetric rates for multiple linear combinations.
  - We provide an interpretation in terms of signal levels for IF in BC.

  - We develop an advanced coding scheme called dirty-paper integer-forcing (DPC-IF).

- Uplink-downlink duality establishment:

  - We establish a sum-rate uplink-downlink duality for IF (without SIC-IF and DPC-IF) under a total power constraint where the same sum rate can be achieved by reversing the roles of transmitters and receivers between MAC and BC.

  - We further establish an advanced sum-rate uplink-downlink duality between SIC-IF and DPC-IF.

- By-products:

  - We propose an iterative optimization algorithm for beamforming and equalization in both the uplink and downlink channel.

  - We show IF (without DPC-IF) can approach the sum capacity of the K-user Gaussian MIMO BC up to a constant gap of $\frac{K}{2}\log K$.

  - We show DPC-IF can achieve the sum capacity of the Gaussian MIMO BC.

1.3.2 Source-channel duality for IF

Similar to the uplink-downlink duality establishment, the source-channel duality establishment for IF is a two-stage process. Modifications will first be performed for integer-forcing source coding in [Ordentlich and Erez, 2013], then we build the duality connection.

- Modification of integer-forcing source coding:

  - We expand [Ordentlich and Erez, 2013] to allow for unequal distortions.

- We generalize [Ordentlich and Erez, 2013] to allow for asymmetric rates across users.
- We develop an advanced technique called successive integer-forcing source coding (SIC-IFSC).

- Source-channel duality establishment:
  - We establish a source-channel duality between SIC-IFSC and SIC-IF where the same rate tuple can be achieved between an IF channel coding problem and its dual distributed source coding problem.
  - We show that even without SIC in both cases, the rates of IF source coding and channel coding lie within a constant gap of one another.

1.3.3 IFIA: Iterative optimization via aligned lattice reduction

For IFIA, this dissertation proposes a class of iterative optimization algorithms. The IF duality results are not directly applied in the algorithms. Instead, the idea of duality inspires and motivates the development of the algorithms for IFIA in this dissertation.

There are two main components: an aligned lattice reduction algorithm and an equalization/beamforming optimization algorithm that utilizes either uplink-downlink duality or convexity. Here we list our contributions as well as comparisons with existing works.

- We consider static channel realizations and the low/moderate SNR regime, whereas [Cadambe and Jafar, 2008, Jafar, 2011] consider the high SNR regime and a large number of channel realizations.
- Prior work [Ntranos et al., 2013b] only considered IFIA for scenarios where the beamforming vectors can be obtained directly from the framework of [Cadambe

and Jafar, 2008]. We propose aligned lattice reduction algorithms to find better beamforming vectors.

- Both [Gomadam et al., 2011] and our work consider limited channel realizations and the low/moderate SNR regime. However, [Gomadam et al., 2011] did not allow for signal scale alignment. The performance of Max-SINR algorithm in [Gomadam et al., 2011] degrades for an infeasible scenario in [Yetis et al., 2010]. We explore both feasible and infeasible scenarios in [Yetis et al., 2010] and show rate gains in both cases.

1.4 Scopes

This dissertation considers constant (block-fading) channels rather than time-varying channels. We will assume all channel matrices are real-valued. Since a complex channel matrix can be transformed to a real-valued channel matrix via its real-valued decomposition, the results in this dissertation can also be applied to complex-valued channel models. We will assume that channel state information (CSI) is available both at the transmitter (CSIT) and receiver (CSIR).

We will make use of the following notation. Column vectors will be denoted by boldface, lowercase font (e.g., $\mathbf{a} \in \mathbb{Z}^L$) and matrices with boldface, uppercase font (e.g., $\mathbf{A} \in \mathbb{Z}^{L \times L}$). Let $\mathbf{a}[i]$ denote the i^{th} coordinate of the vector $\mathbf{a}$. We will use $\|\mathbf{a}\|$ to represent ℓ_2-norm of $\mathbf{a}$ and $\text{Tr}(\mathbf{A})$ to represent the trace of $\mathbf{A}$. We will also use $\text{diag}(\mathbf{a})$ to denote the diagonal matrix formed by using the placing the elements of $\mathbf{a}$ along the diagonal. All logarithms are taken to base 2 and we define $\log^+(x) \triangleq \max(0, \log x)$. We denote the identity matrix by $\mathbf{I}$, the all-ones column vector of length k by $\mathbf{1}_k$ and the all-zeros column vector of length k by $\mathbf{0}_k$. Let $\mathcal{P}_\pi$ denote the permutation matrix corresponding to permutation π.

We will work with both the real field $\mathbb{R}$ and prime-sized finite fields

$\mathbb{Z}_p = \{0, 1, \ \ldots \ , p-1\}$ where p is prime. We will denote addition and summation over the reals by $+$ and $\sum$, respectively. For a prime-sized finite field, we will use $\oplus$ and $\bigoplus$ to denote addition and summation, respectively. Define $[a] \bmod p$ to be the modulo-p reduction of a. For vectors and matrices , the modulo-p operation is taken elementwise and denoted by $[\mathbf{a}] \bmod p$ and $[\mathbf{A}] \bmod p$, respectively. Taking a linear combination over a prime-sized finite field can be linked to taking a linear combination over the reals as follows,

$$q_1 w_1 \oplus q_2 w_2 = [q_1 w_1 + q_2 w_2] \bmod p \ . \tag{1.1}$$

Note that, on the left-hand side, q_1, q_2, w_1, w_2 are elements of the finite field whereas, on the right-hand side, they are elements of the integers under the natural mapping. Finally, subscripts "u" and "d" will be used to denote variables associated with the uplink and downlink, respectively.

The remainder of this dissertation is organized as follows. We first review some basic lattice definitions in Chapter 2. In Chapter 3 we include the problem statement, proposed coding scheme and achievable rates for IF for the MIMO MAC. We do the same for IF broadcast channel coding in Chapter 4 and IF source coding in Chapter 5. In Chapter 6, we establish both uplink-downlink duality and channel-source duality for IF. We also include by-products for IF uplink-downlink duality and a constant gap result for IF channel-source duality. We switch to the IFIA problem in Chapter 7 including a problem statement, algorithms and simulations. Finally, we state our conclusions and summarize continuing works in Chapter 8.

Chapter 2

Lattice Preliminaries

2.1 Lattice Definitions

Below, we review some basic lattice definitions as well as nested lattice code constructions that we will need for our achievability results. See [Zamir, 2014] for a thorough introduction to lattice codes.

A *lattice* Λ is a discrete additive subgroup of $\mathbb{R}^n$ such that, if $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \Lambda$, then $\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \in \Lambda$ and $-\boldsymbol{\lambda}_1, -\boldsymbol{\lambda}_2 \in \Lambda$. The *nearest neighbor quantizer* associated to Λ is defined as

$$Q_\Lambda(\mathbf{x}) \triangleq \arg\min_{\boldsymbol{\lambda} \in \Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\| \tag{2.1}$$

(with ties broken in a systematic fashion). The *fundamental Voronoi region* $\mathcal{V}$ of Λ is the set of all points in $\mathbb{R}^n$ that quantize to $\mathbf{0}$. We define the *second moment* Λ as

$$\sigma^2(\Lambda) \triangleq \frac{1}{n} \int_{\mathcal{V}} \|\mathbf{x}\|^2 \frac{1}{\mathrm{Vol}(\mathcal{V})} d\mathbf{x} \tag{2.2}$$

where $\mathrm{Vol}(\mathcal{V})$ denotes the volume of $\mathcal{V}$.

We also define the *modulo operation* with respect to Λ as

$$[\mathbf{x}] \bmod \Lambda \triangleq \mathbf{x} - Q_\Lambda(\mathbf{x}) \tag{2.3}$$

and note that it satisfies a distributive law, $[a[\mathbf{x}] \bmod \Lambda + b\mathbf{y}] \bmod \Lambda = [a\mathbf{x} + b\mathbf{y}] \bmod \Lambda$ for all $a, b \in \mathbb{Z}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Lemma 1 (Crypto Lemma)**.** *Let* $\mathbf{x}$ *be a random vector over* $\mathbb{R}^n$ *and* $\mathbf{d}$ *be an independent random vector drawn uniformly over the Voronoi region* $\mathcal{V}$ *of the lattice* Λ*. The modulo sum* $[\mathbf{x}+\mathbf{d}] \bmod \Lambda$ *is independent of* $\mathbf{x}$ *and uniform over* $\mathcal{V}$*.*

See [Zamir, 2014, Ch 4.1] for a full proof.

The lattice Λ_{C} is said to be *nested* in the lattice Λ_F if $\Lambda_{\mathrm{C}} \subset \Lambda_{\mathrm{F}}$. In this case, Λ_{C} is called the coarse lattice and Λ_{F} the fine lattice. A *nested lattice codebook* $\mathcal{L} = \Lambda_{\mathrm{F}} \cap \mathcal{V}_{\mathrm{C}}$ consists of all fine lattice points that fall in the fundamental Voronoi region $\mathcal{V}_{\mathrm{C}}$ of the coarse lattice. Note that nested lattices satisfy the following quantization property:

$$\Big[Q_{\Lambda_{\mathrm{F}}}(\mathbf{x})\Big] \bmod \Lambda_{\mathrm{C}} = \Big[Q_{\Lambda_{\mathrm{F}}}([\mathbf{x}] \bmod \Lambda_{\mathrm{C}})\Big] \bmod \Lambda_{\mathrm{C}} \ . \tag{2.4}$$

2.2 Nested Lattice Codes and Properties

Our encoding strategies rely on the existence of good nested lattice codebooks. Below, we describe the nested lattice ensemble as well as properties that are central to our achievability proofs. Our notation closely follows that from [Nazer et al., 2016, §IV], which contains a more detailed exposition.

Recall that n denotes the blocklength of our coding scheme. Let p represent a prime number and $\mathbb{Z}_p$ the finite field of size p. We will also need integer-valued parameters $0 \leq k_{\mathrm{C},\ell} \leq k_{\mathrm{F},\ell},\ \ell = 1, \ldots, L$. Define $k_{\mathrm{C}} \triangleq \min_\ell k_{\mathrm{C},\ell}$, $k_{\mathrm{F}} \triangleq \max_\ell k_{\mathrm{F},\ell}$, and $k \triangleq k_{\mathrm{F}} - k_{\mathrm{C}}$.

The construction begins with the generator matrix of a linear code $\mathbf{G} \in \mathbb{Z}_p^{k_{\mathrm{F}} \times n}$. For $\ell = 1, \ldots, L$, define $\mathbf{G}_{\mathrm{C},\ell}$ and $\mathbf{G}_{\mathrm{F},\ell}$ to be the submatrices consisting of the first $k_{\mathrm{C},\ell}$ and $k_{\mathrm{F},\ell}$ rows of $\mathbf{G}$, respectively. Let

$$\mathcal{C}_{\mathrm{C},\ell} = \left\{ \mathbf{G}_{\mathrm{C},\ell}^{\mathsf{T}} \mathbf{w} : \mathbf{w} \in \mathbb{Z}_p^{k_{\mathrm{C},\ell}} \right\} \tag{2.5}$$

$$\mathcal{C}_{\mathrm{F},\ell} = \left\{ \mathbf{G}_{\mathrm{F},\ell}^{\mathsf{T}} \mathbf{w} : \mathbf{w} \in \mathbb{Z}_p^{k_{\mathrm{F},\ell}} \right\} \tag{2.6}$$

denote the resulting linear codebooks. For $\gamma > 0$ to be specified later, define the mapping $\phi(w) \triangleq \gamma p^{-1} w$ from $\mathbb{Z}_p$ to $\mathbb{R}$. We also define the inverse mapping $\bar{\phi}(\kappa) \triangleq [\gamma^{-1} p \kappa] \bmod p$, which is only defined on the domain $\gamma p^{-1}\mathbb{Z}$. Both of these mappings are taken elementwise when applied to vectors and will be used to go back and forth between linear codebooks and lattices.

We now generate L coarse lattices and L fine lattices as follows:

$$\Lambda_{\mathrm{C},\ell} = \left\{ \boldsymbol{\lambda} \in \gamma p^{-1} \mathbb{Z}^n : \bar{\phi}(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathrm{C},\ell} \right\} \tag{2.7}$$

$$\Lambda_{\mathrm{F},\ell} = \left\{ \boldsymbol{\lambda} \in \gamma p^{-1} \mathbb{Z}^n : \bar{\phi}(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathrm{F},\ell} \right\} . \tag{2.8}$$

By construction, these lattices are nested according to the order for which the parameters $k_{\mathrm{C},\ell}$ and $k_{\mathrm{F},\ell}$ are increasing. Define Λ_{C} and Λ_{F} to be the coarsest and finest lattices in the ensemble, respectively. Let $\mathcal{V}_{\mathrm{C},\ell}$ and $\mathcal{V}_{\mathrm{F},\ell}$ denote the Voronoi regions of $\Lambda_{\mathrm{C},\ell}$ and $\Lambda_{\mathrm{F},\ell}$, respectively. Finally, we take the elements of the fine lattice $\Lambda_{\mathrm{F},\ell}$ that fall in the Voronoi region of the coarse lattice $\Lambda_{\mathrm{C},\ell}$ to be the nested lattice codebook

$$\mathcal{L}_\ell \triangleq \Lambda_{\mathrm{F},\ell} \cap \mathcal{V}_{\mathrm{C},\ell} \tag{2.9}$$

for the ℓ^{th} user.

The theorem below summarizes results from [Ordentlich and Erez, 2015b] that demonstrate that this nested lattice construction exhibits good shaping and noise tolerance properties.

Theorem 1 ([Ordentlich and Erez, 2015b, Theorem 2])**.** *For $\ell = 1, \ldots, L$, select parameters $P_\ell > 0$ and $0 < \sigma^2_{\text{eff},\ell} < P_\ell$. Then, for any $\epsilon > 0$ and n and p large enough, there are parameters γ, $k_{\mathrm{C},\ell}$, and $k_{\mathrm{F},\ell}$ and a generator matrix $\mathbf{G} \in \mathbb{Z}_p^{k_{\mathrm{F}} \times n}$ such that, for $\ell = 1, \ldots, L$*

(a) the submatrices $\mathbf{G}_{\mathrm{C},\ell}$ and $\mathbf{G}_{\mathrm{F},\ell}$ are full rank.

(b) the coarse lattices $\Lambda_{\mathrm{C},\ell}$ have second moments close to their power constraints

$$P_\ell - \epsilon < \sigma^2(\Lambda_{\mathrm{C},\ell}) < P_\ell \ .$$

(c) the lattices can tolerate the desired level of effective noise. Let $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_L$ be independent noise vectors where $\mathbf{z}_0 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{z}_\ell \sim \mathrm{Unif}(\mathcal{V}_{\mathrm{C},\ell})$. For any $\beta_0, \beta_1, \ldots, \beta_L \in \mathbb{R}$, let $\mathbf{z}_{\text{eff}} = \sum_{\ell=0}^{L} \mathbf{z}_\ell$. If $\beta_0^2 + \sum_{\ell=1}^{L} \beta_\ell^2 P_\ell \leq \sigma^2_{\text{eff},m}$, any fine lattice point $\boldsymbol{\lambda} \in \Lambda_{\mathrm{F},m}$ can recover from $\mathbf{z}_{\text{eff}}$ with high probability,

$$\mathbb{P}\Big(Q_{\Lambda_{\mathrm{F},m}}(\boldsymbol{\lambda} + \mathbf{z}_{\text{eff}}) \neq \boldsymbol{\lambda} \Big) < \epsilon \ .$$

Similarly, if $\beta_0^2 + \sum_{\ell=1}^{L} \beta_\ell^2 P_\ell \leq P_m$, any coarse lattice point $\boldsymbol{\lambda} \in \Lambda_{\mathrm{C},m}$ can recover from $\mathbf{z}_{\text{eff}}$ with high probability,

$$\mathbb{P}\Big(Q_{\Lambda_{\mathrm{C},m}}(\boldsymbol{\lambda} + \mathbf{z}_{\text{eff}}) \neq \boldsymbol{\lambda} \Big) < \epsilon \ .$$

(d) the rates of the nested lattice codes satisfy

$$\frac{1}{n} \log |\mathcal{L}_\ell| = \frac{k_{\mathrm{F},\ell} - k_{\mathrm{C},\ell}}{n} \log_2 p > \frac{1}{2} \log \left(\frac{P_\ell}{\sigma^2_{\text{eff},\ell}} \right) - \epsilon \ .$$

Finally, it can be argued that we can label lattice codewords so that integer-linear combinations of codewords correspond to linear combinations of the messages over $\mathbb{Z}_p$. We recall the definition of a linear labeling from [Feng et al., 2013].

Definition 1. *We say that a mapping $\varphi : \Lambda_{\mathrm{F}} \to \mathbb{Z}_p^k$ is a linear labeling if*

(a) $\boldsymbol{\lambda} \in \Lambda_{\mathrm{F},\ell}$ if and only if the last $k_{\mathrm{F}} - k_{\mathrm{F},\ell}$ components of its label $\varphi(\boldsymbol{\lambda})$ are zero. Similarly, $\boldsymbol{\lambda} \in \Lambda_{\mathrm{C},\ell}$ if and only if the last $k_{\mathrm{F}} - k_{\mathrm{C},\ell}$ components of its label $\varphi(\boldsymbol{\lambda})$ are zero.

(b) For all $a_\ell \in \mathbb{Z}$ and $\boldsymbol{\lambda}_\ell \in \Lambda_{\mathrm{F}}$, we have that

$$\varphi\left(\sum_{\ell=1}^{L} a_\ell \boldsymbol{\lambda}_\ell \right) = \bigoplus_{\ell=1}^{L} q_\ell \varphi(\boldsymbol{\lambda}_\ell)$$

where $q_\ell = [a_\ell] \bmod p$.

Consider the mapping that sets $\varphi(\boldsymbol{\lambda})$ to be the last k components of the unique vector $\mathbf{v} \in \mathbb{Z}_p^{k_{\mathrm{F}}}$ satisfying $\bar{\phi}(\boldsymbol{\lambda}) = \mathbf{G}^{\mathsf{T}}\mathbf{v}$. From [Nazer et al., 2016, Theorem 10], φ is a linear labeling. We also define the inverse map

$$\bar{\varphi} \triangleq \phi\left(\mathbf{G}^{\mathsf{T}} \begin{bmatrix} \mathbf{0}_{k_{\mathrm{C}}} \\ \mathbf{w} \end{bmatrix}\right) ,$$

which satisfies $\varphi\Big(\bar{\varphi}(\mathbf{w})\Big) = \mathbf{w}$.

2.3 Nested Lattices to Signal Levels

For a given message $\mathbf{w}_{\mathrm{u},\ell} \in \mathbb{Z}_p^{k_{\mathrm{F},\ell} - k_{\mathrm{C},\ell}}$ where $k_{\mathrm{F},\ell}$ and $k_{\mathrm{C},\ell}$ are integers representing signal levels, we select a shaping lattice $\Lambda_{\mathrm{C},\ell}$ related to the power constraint and a coding lattice $\Lambda_{\mathrm{F},\ell}$ related to the noise tolerance. All lattice codebooks are subsets of $\mathcal{L} \triangleq \Lambda_{\mathrm{F,max}} \cap \mathcal{V}_{\mathrm{C,min}}$ where $\Lambda_{\mathrm{F,max}}$ is the finest coding lattice and $\Lambda_{\mathrm{C,min}}$ is the coarsest shaping lattice such that $\Lambda_{\mathrm{C,min}} \subset \Lambda_{\mathrm{C},\ell} \subset \Lambda_{\mathrm{F},\ell} \subset \Lambda_{\mathrm{F,max}}$ for all ℓ. The higher coding power we have, the smaller $k_{\mathrm{C},\ell}$ will be and the coarser the $\Lambda_{\mathrm{C},\ell}$ we will choose. The smaller noise tolerance is, the smaller $k_{\mathrm{F},\ell}$ will be and the finer coding lattice $\Lambda_{\mathrm{F},\ell}$ we will select. The message $\mathbf{w}_{\mathrm{u},\ell}$ will be encoded using codebook $\Lambda_{\mathrm{F},\ell} \cap \mathcal{V}_{\mathrm{C},\ell}$. Fig 2·1 shows the relationship between signal levels and the nested lattice code. Message $\mathbf{w}_{\mathrm{u},\ell}$ lies below the power levels related to the shaping lattice $\Lambda_{\mathrm{C},\ell}$ and above noise tolerance level related to the coding lattice $\Lambda_{\mathrm{F},\ell}$. For details about the signal level representation of nested lattice codes, we refer readers to [Nazer et al., 2016].

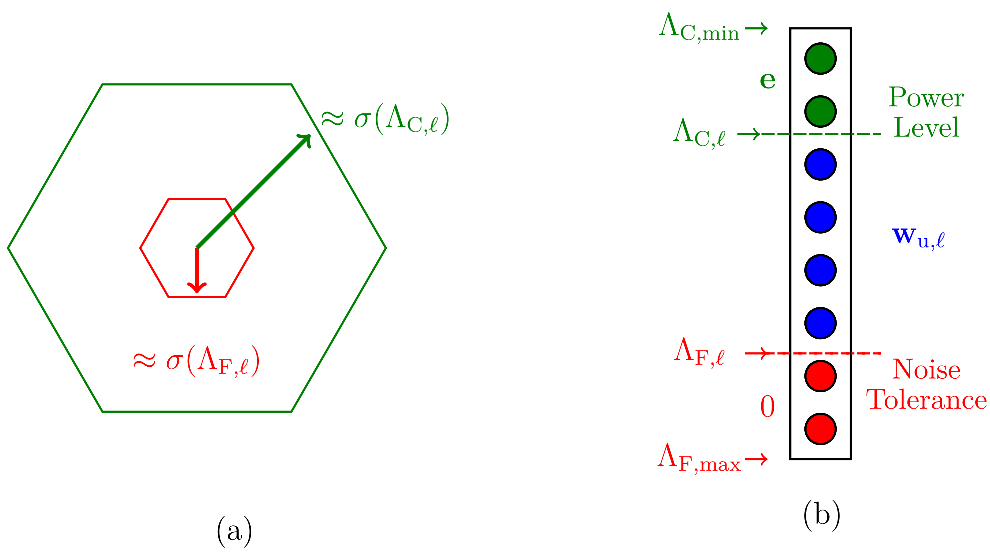

Figure 2·1: (a) Nested lattices used to encode $\mathbf{w}_{\mathrm{u},\ell}$; and (b) Signal levels correspond to nested lattice. Here $k_{\mathrm{F},\ell} - k_{\mathrm{C},\ell} = 4$ is the number of signal levels allocated to message $\mathbf{w}_{\mathrm{u},\ell}$. Signal levels below noise tolerance will be assigned to 0. Signal levels above power level are "don't care" entries $\mathbf{e}$ that can take any value. For unequal power and asymmetric rate, different messages will have different power levels and noise tolerances, thus lead to different shaping lattices $\Lambda_{\mathrm{C},\ell}$ and coding lattices $\Lambda_{\mathrm{F},\ell}$. All shaping lattices and coding lattices are nested in a certain order and can be constructed using the codebook $\mathcal{L} \triangleq \Lambda_{\mathrm{F,max}} \cap \mathcal{V}_{\mathrm{C,min}}$.

Chapter 3

IF in MIMO Multiple-access Channel Coding (Uplink)

3.1 Uplink Architecture

3.1.1 Problem Statement

The uplink channel (i.e., MIMO MAC) consists of L transmitters and a single N-antenna receiver. The ℓ^{th} transmitter is equipped with M_ℓ transmit antennas. It has a *message* $w_{\mathrm{u},\ell}$ that is drawn independently and uniformly from $\{1, 2, \ldots, 2^{nR_{\mathrm{u},\ell}}\}$[1] and an *encoder* $\mathcal{E}_{\mathrm{u},\ell} : \{1, 2, \ldots, 2^{nR_{\mathrm{u},\ell}}\} \to \mathbb{R}^{M_\ell \times n}$ that maps this message into a *channel input* $\mathbf{X}_{\mathrm{u},\ell} = \mathcal{E}_{\mathrm{u},\ell}(w_{\mathrm{u},\ell})$ of blocklength n. It will often be convenient to work with the concatenation of the channel inputs

$$\mathbf{X}_{\mathrm{u}} \triangleq \begin{bmatrix} \mathbf{X}_{\mathrm{u},1} \\ \vdots \\ \mathbf{X}_{\mathrm{u},L} \end{bmatrix} \tag{3.1}$$

which is of dimension $M \times n$ where $M = \sum_\ell M_\ell$ denotes the total number of transmit antennas. Conventional MAC models impose a power constraint on each user individually. However, it is well-known that uplink-downlink duality can be established only if we are free to reallocate the power across transmitters [Vishwanath et al., 2003, Viswanath and Tse, 2003, Yu and Cioffi, 2004]. In this dissertation, the transmitters must satisfy a *total power constraint* $\mathbb{E}\Big[\operatorname{Tr}(\mathbf{X}_{\mathrm{u}}\mathbf{X}_{\mathrm{u}}^{\mathsf{T}})\Big] \le nP_{\text{total}}$.

[1] $R_{\mathrm{u},\ell}$ is the coding rate for the ℓ^{th} transmitter.

The receiver observes a *noisy linear combination* of the emitted signals,

$$\mathbf{Y}_{\mathrm{u}} = \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{u},\ell}\mathbf{X}_{\mathrm{u},\ell} + \mathbf{Z}_{\mathrm{u}} \tag{3.2}$$

where $\mathbf{H}_{\mathrm{u},\ell} \in \mathbb{R}^{N \times M_\ell}$ is the *channel matrix* from the ℓ^{th} transmitter to the receiver and the additive noise $\mathbf{Z}_{\mathrm{u}} \in \mathbb{R}^{N \times n}$ is elementwise i.i.d. Gaussian with mean zero and variance one. We denote the concatenated channel matrices by

$$\mathbf{H}_{\mathrm{u}} \triangleq \left[\mathbf{H}_{\mathrm{u},1} \ \cdots \ \mathbf{H}_{\mathrm{u},L}\right] , \tag{3.3}$$

which lets us concisely write the channel output as

$$\mathbf{Y}_{\mathrm{u}} = \mathbf{H}_{\mathrm{u}}\mathbf{X}_{\mathrm{u}} + \mathbf{Z}_{\mathrm{u}} \ . \tag{3.4}$$

This channel output is sent through a *decoder* $\mathcal{D}_{\mathrm{u}} : \mathbb{R}^{N \times n} \to \{1, 2, \ldots, 2^{nR_1}\} \times \cdots \times \{1, 2, \ldots, 2^{nR_L}\}$ that produces estimates of the messages, $(\hat{w}_{\mathrm{u},1}, \ldots, \hat{w}_{\mathrm{u},L}) = \mathcal{D}_{\mathrm{u}}(\mathbf{Y}_{\mathrm{u}})$.

Overall, we say that the uplink rates $R_{\mathrm{u},1}, \ldots, R_{\mathrm{u},L}$ are *achievable* if, for any $\epsilon > 0$ and n large enough, there exist encoders and decoder such that $\mathbb{P}\Big(\bigcup_{\ell=1}^{L}\{\hat{w}_{\mathrm{u},\ell} \neq w_{\mathrm{u},\ell}\}\Big) < \epsilon$. The uplink capacity region is the closure of the set of all achievable rates.

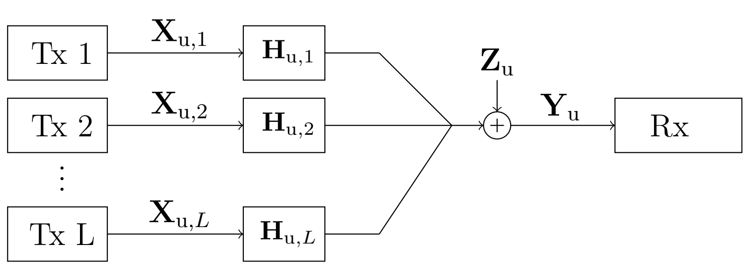

Figure 3·1: Block diagram of the uplink channel models.

3.1.2 Conventional Approach: ZF Linear Receiver

The ℓ^{th} transmitter has a codeword $\mathbf{s}_{\text{u},\ell} \in \mathbb{R}^n$ with expected power $\frac{1}{n}\mathbb{E}\|\mathbf{s}_{\text{u},\ell}\|^2 = P_{\text{u},\ell}$. It uses a beamforming vector $\mathbf{c}_\ell \in \mathbb{R}^{M_\ell}$ to generate its channel input

$$\mathbf{X}_{\text{u},\ell} = \mathbf{c}_{\text{u},\ell}\mathbf{s}_{\text{u},\ell}^{\mathsf{T}} \ . \tag{3.5}$$

Collecting the beamforming vectors into the matrix

$$\mathbf{C}_{\text{u}} \triangleq \begin{bmatrix} \mathbf{c}_{\text{u},1} & \mathbf{0}_{M_1} & \dots & \mathbf{0}_{M_1} \\ \mathbf{0}_{M_2} & \mathbf{c}_{\text{u},2} & \dots & \mathbf{0}_{M_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{M_L} & \mathbf{0}_{M_L} & \dots & \mathbf{c}_{\text{u},L} \end{bmatrix} \tag{3.6}$$

and the codewords into the matrix

$$\mathbf{S}_{\text{u}} \triangleq \begin{bmatrix} \mathbf{s}_{\text{u},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{s}_{\text{u},L}^{\mathsf{T}} \end{bmatrix} \tag{3.7}$$

we can write the beamforming operation as

$$\mathbf{X}_{\text{u}} = \mathbf{C}_{\text{u}}\mathbf{S}_{\text{u}} \ . \tag{3.8}$$

To recover the m^{th} codeword, the receiver uses an equalization vector $\mathbf{b}_{\text{u},m} \in \mathbb{R}^N$ to obtain the effective channel output

$$\tilde{\mathbf{y}}_{\text{u},m}^{\mathsf{T}} = \mathbf{b}_{\text{u},m}^{\mathsf{T}}\mathbf{Y}_{\text{u}} \tag{3.9}$$

$$= \underbrace{\mathbf{b}_{\text{u},m}^{\mathsf{T}}\mathbf{H}_{\text{u},m}\mathbf{c}_{\text{u},m}\mathbf{s}_{\text{u},m}^{\mathsf{T}}}_{\text{signal}} + \underbrace{\sum_{\ell \neq m} \mathbf{b}_{\text{u},m}^{\mathsf{T}}\mathbf{H}_{\text{u},\ell}\mathbf{c}_{\text{u},\ell}\mathbf{s}_{\text{u},\ell}^{\mathsf{T}}}_{\text{interference}} + \underbrace{\mathbf{b}_{\text{u},m}^{\mathsf{T}}\mathbf{Z}_{\text{u}}}_{\text{noise}} \ , \tag{3.10}$$

which is fed into a single-user decoder. By employing i.i.d. Gaussian codewords, the transmitters can achieve the following rates

$$R_{\mathrm{u},m} = \frac{1}{2}\log\left(1 + \frac{P_{\mathrm{u},m}\left|\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{u},m}\mathbf{c}_{\mathrm{u},m}\right|^2}{\sum_{\ell\neq m} P_{\mathrm{u},\ell}\left|\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{u},\ell}\mathbf{c}_{\mathrm{u},\ell}\right|^2}\right) \quad m = 1, \ \ldots \ , L \ . \tag{3.11}$$

3.1.3 Capacity Region for Uplink MIMO MAC

The uplink (i.e., MIMO MAC) capacity region $\mathcal{C}_{\mathrm{u}}$ is the set of rate tuples $(R_{\mathrm{u},1}, \ldots, R_{\mathrm{u},L})$ satisfying

$$\sum_{\ell\in\mathcal{S}} R_\ell \leq \frac{1}{2}\log\det\left(\mathbf{I} + \sum_{\ell\in\mathcal{S}} \mathbf{H}_{\mathrm{u},\ell}\mathbf{K}_\ell\mathbf{H}_{\mathrm{u},\ell}^{\mathsf{T}}\right) \quad \forall\mathcal{S}\subseteq\{1,2,\ldots,L\} \tag{3.12}$$

for some positive semi-definite matrices $\mathbf{K}_1, \ldots, \mathbf{K}_L$ satisfying the sum power constraint $\sum_{\ell=1}^{L}\mathrm{Tr}(\mathbf{K}_\ell) \leq P_{\mathrm{total}}$. It can be attained with i.i.d. Gaussian encoding and simultaneous joint typicality decoding. See [El Gamal and Kim, 2011, section 9.2.1] for more details.

3.2 Uplink Integer-Forcing Architecture

Our uplink coding scheme is taken from [Nazer et al., 2016, Section VI]. Below, we summarize the encoding and decoding operations for successive integer-forcing in order to highlight the similarities between the uplink and downlink integer-forcing schemes.

We begin by selecting a power allocation $\mathbf{P}_{\mathrm{u}} = \mathrm{diag}(P_{\mathrm{u},1}, \ldots, P_{\mathrm{u},L})$ for the codewords and a beamforming matrix $\mathbf{C}_{\mathrm{u}}$ satisfying (3.6). Note that, in order to meet the total power constraint with equality, we require that $\mathrm{Tr}(\mathbf{C}_{\mathrm{u}}^{\mathsf{T}}\mathbf{C}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}) = P_{\mathrm{total}}$. We also select a full-rank integer matrix $\mathbf{A}_{\mathrm{u}} \in \mathbb{Z}^{L\times L}$, an equalization matrix $\mathbf{B}_{\mathrm{u}} = [\mathbf{b}_{\mathrm{u},1} \ \cdots \ \mathbf{b}_{\mathrm{u},L}]^{\mathsf{T}} \in \mathbb{R}^{L\times N}$, and an $L \times L$ lower unitriangular[2] successive cancellation matrix $\mathbf{R}_{\mathrm{u}}$. These choices specify the effective noise variances $\sigma^2_{\mathrm{u,SIC},m}$ which

[2] A unitriangular matrix is a triangular matrix with unit entries along its diagonal.

will be introduced later.

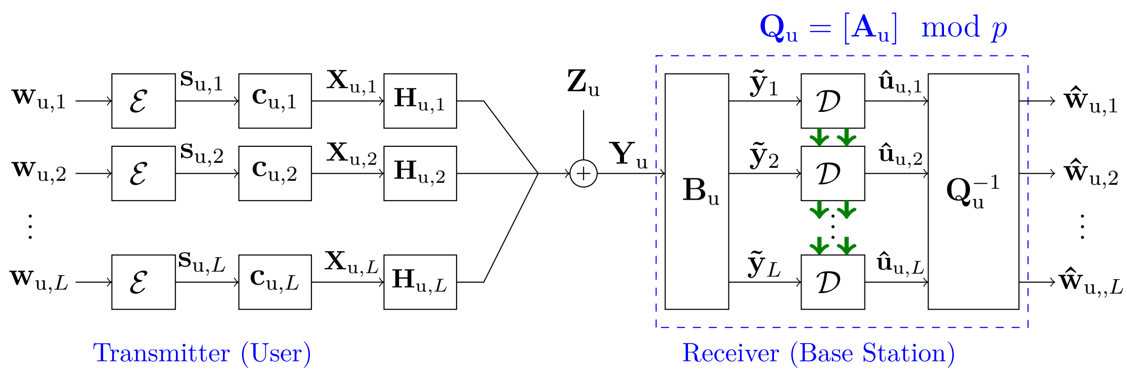

Figure 3·2: Block diagram of the integer-forcing uplink architecture. Here $\mathcal{E}$ and $\mathcal{D}$ represent SISO encoder and decoder, respectively. Each message vector $\mathbf{w}_{\mathrm{u},\ell}$ is encoded into a dithered lattice codeword $\mathbf{s}_{\mathrm{u},\ell}$ and mapped to a channel input $\mathbf{X}_{\mathrm{u},\ell} = \mathbf{c}_{\mathrm{u},\ell}\mathbf{s}_{\mathrm{u},\ell}^{\mathsf{T}}$. For $m = 1, \ldots, L$, the receiver uses an equalized channel output $\tilde{\mathbf{y}}_{\mathrm{u},m} = \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{Y}_\mathrm{u}$ to make an estimate $\hat{\mathbf{u}}_{\mathrm{u},m}$ of the linear combination $\mathbf{u}_{\mathrm{u},m}$. The SISO decoders are potentially enhanced by successive cancellation (illustrated with green arrows). Finally, the linear combinations are inverted to recover estimates $\hat{\mathbf{w}}_{\mathrm{u},\ell}$ of the message vectors.

3.2.1 Overview of Uplink IF without SIC

The operations at the transmitters are the same as ZF, except that we use a nested lattice codebook to ensure that integer-linear combinations of codewords are themselves codewords. The goal is to recover L integer-linear combinations of the form $\mathbf{a}_{\mathrm{u},1}^{\mathsf{T}}\mathbf{S}_\mathrm{u}, \ldots, \mathbf{a}_{\mathrm{u},L}^{\mathsf{T}}\mathbf{S}_\mathrm{u}$ where the $\mathbf{a}_{\mathrm{u},m}^{\mathsf{T}}$ are the rows of a full-rank integer matrix $\mathbf{A}_\mathrm{u} \in \mathbb{Z}^{L\times L}$, i.e.,

$$\mathbf{A}_\mathrm{u} = \begin{bmatrix} \mathbf{a}_{\mathrm{u},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_{\mathrm{u},L}^{\mathsf{T}} \end{bmatrix} . \tag{3.13}$$

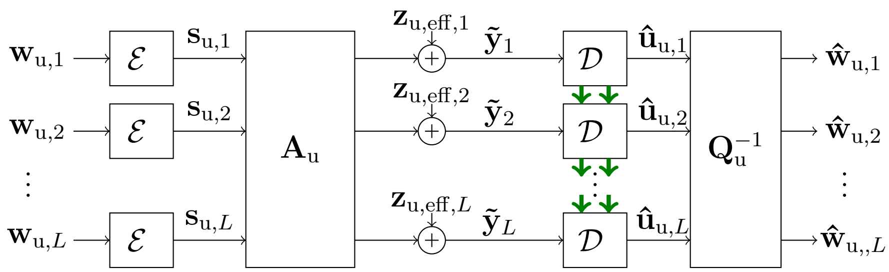

Figure 3·3: Block diagram of the effective channel induced by the integer-forcing uplink architecture. The m^{th} decoder observes an integer-linear combination of the codewords plus effective noise, $\sum_{\ell} a_{\mathrm{u},m,\ell}\mathbf{s}_{\mathrm{u},\ell} + \mathbf{z}_{\mathrm{u,eff},m}$ from which it makes an estimate of the linear combination $\mathbf{u}_{\mathrm{u},\ell}$ with coefficients $q_{\mathrm{u},m,\ell} = [a_{\mathrm{u},m,\ell}] \bmod p$. (If the decoders use successive interference cancellation, then $\mathbf{z}_{\mathrm{u,eff},m}$ is replaced with $\mathbf{z}_{\mathrm{u,SIC},m}$.) Finally, it applies the inverse of the matrix $\mathbf{Q}_{\mathrm{u}} = \{q_{\mathrm{u},m,\ell}\}$ over $\mathbb{Z}_p$ to estimate the message.

To recover the m^{th} linear combination $\mathbf{a}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{u}}$, the receiver applies an equalization vector $\mathbf{b}_{\mathrm{u},m} \in \mathbb{R}^{M_m}$ to form the effective channel output

$$\tilde{\mathbf{y}}_{\mathrm{u},m}^{\mathsf{T}} = \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{u}} \tag{3.14}$$

$$= \mathbf{a}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{u}} + \mathbf{z}_{\mathrm{u,eff},m}^{\mathsf{T}} \tag{3.15}$$

$$\mathbf{z}_{\mathrm{u,eff},m}^{\mathsf{T}} \triangleq \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{Z}_{\mathrm{u}} + \left(\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}} - \mathbf{a}_{\mathrm{u},m}^{\mathsf{T}}\right)\mathbf{S}_{\mathrm{u}} \ . \tag{3.16}$$

We define the *effective noise variance* as

$$\sigma_{\mathrm{u,eff},m}^2 \triangleq \frac{1}{n}\mathbb{E}\|\mathbf{z}_{\mathrm{u,eff},m}\|^2 \tag{3.17}$$

$$= \|\mathbf{b}_{\mathrm{u},m}\|^2 + \left\|\left(\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}} - \mathbf{a}_{\mathrm{u},m}^{\mathsf{T}}\right)\mathbf{P}_{\mathrm{u}}^{1/2}\right\|^2 \ . \tag{3.18}$$

The structure of the integer matrix $\mathbf{A}_{\mathrm{u}}$ determines which codewords can be canceled out in each decoding step. In order to keep our notation manageable, we assume that $\mathbf{A}_{\mathrm{u}}$ is selected so that the m^{th} user can be associated with the m^{th} effective noise variance. The following definition describes when this is possible.

Definition 2. *We say that the identity permutation is admissible for a full-rank integer matrix* $\mathbf{A}_{\mathrm{u}}$ *if*

(a) *the effective noise variances are in increasing order,* $\sigma^2_{\mathrm{u,eff},1} \leq \cdots \leq \sigma^2_{\mathrm{u,eff},,L}$ *and*

(b) *there exists a lower unitriangular matrix* $\mathbf{L} \in \mathbb{R}^{L\times L}$ *such that* $\mathbf{L}\mathbf{A}_{\mathrm{u}}$ *is upper triangular.*

The first condition can always be met by reordering the rows of $\mathbf{A}_{\mathrm{u}}$. The second condition always holds up to a column permutation on $\mathbf{A}_{\mathrm{u}}$.

Using the *algebraic successive cancellation* technique introduced by Ordentlich *et al.* in [Ordentlich et al., 2014], we can assign each effective noise variance to a single transmitter.[3] Interested readers can find details in [Ordentlich et al., 2014, Nazer et al., 2016]. We will include detailed discussion of algebraic successive cancellation in Section 3.2.3. Here, we will assume that the transmitters have been reindexed so that the identity permutation is admissible according to Definition 2.

It follows from [Nazer et al., 2016, Lemma 10] that the following rates (without SIC, which is a special case for the SIC scheme in Section 3.2.3) are achievable:

$$R_{\mathrm{u},m} = \frac{1}{2}\log^+\left(\frac{P_{\mathrm{u},m}}{\sigma^2_{\mathrm{u,eff},m}}\right) \quad m = 1, \ldots, L \; . \tag{3.19}$$

To recover the codewords, the receiver now just applies the inverse of the integer matrix. As argued in [Zhan et al., 2014], this inverse can be performed over the finite field from which the messages and nested lattice codes are drawn.

Even without SIC, IF shows its advantage to ZF in both transmission rate and robustness to imperfect channel state information. Figure 3·4 and Figure 3·5 show the advantages with comparisons to ZF and channel capacity. In Figure 3·4, we assume symmetric coding powers and asymmetric rates. We compare the performance of IF, ZF and joint ML decoder by averaging the asymmetric rates among 4 users. In

[3] Without algebraic successive cancellation, all transmitters will be constrained by the worst effective noise variance, which will prevent us from establishing uplink-downlink duality.

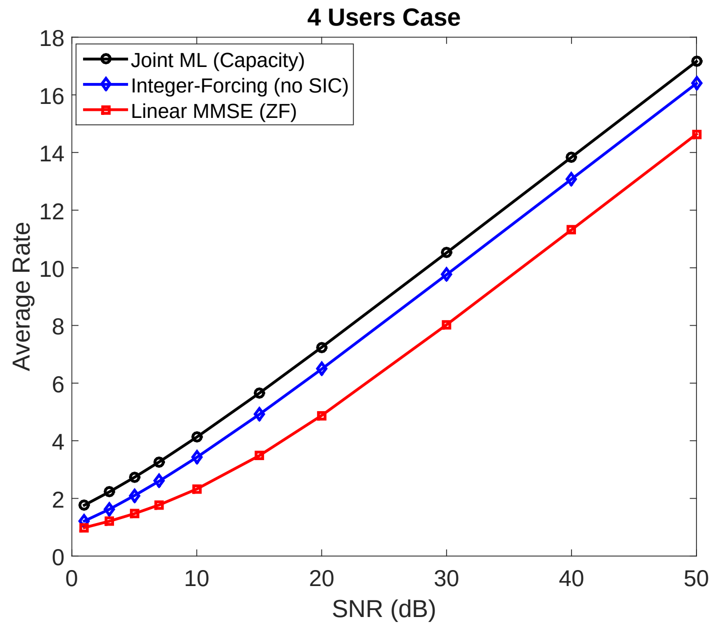

Figure 3·4: Comparison of average rate between IF, ZF and Capacity. Here we compare the average rate (bits per transmission) for 4 users in the uplink.

Figure 3·5, we assume the receiver observes a noisy channel matrix $\mathbf{H}_{\mathrm{u}} + \mathbf{Z}_{\mathrm{c,u}}$ where $\mathbf{H}_{\mathrm{u}}$ is the channel matrix and $\mathbf{Z}_{\mathrm{c,u}}$ is i.i.d. Gaussian noise matrix with zero means. We increase the noise variance of $\mathbf{Z}_{\mathrm{c,u}}$ and compare the performance of IF and ZF by averaging the 1% outage rate across 4 users.

3.2.2 Overview of Uplink IF with SIC

The performance of uplink integer-forcing architecture can also be improved by the SIC technique. For the uplink, each decoder can use recovered integer-linear combinations as side information to improve its effective channel [Ordentlich et al., 2013]. Let $\mathbf{R}_{\mathrm{u}}$ be a lower, unitriangular matrix. At the m^{th} decoder, we assume that $\mathbf{a}_{\mathrm{u},1}^{\mathsf{T}}\mathbf{S}_{\mathrm{u}}, \ldots, \mathbf{a}_{\mathrm{u},m-1}^{\mathsf{T}}\mathbf{S}_{\mathrm{u}}$ have been recovered correctly and form the following effective

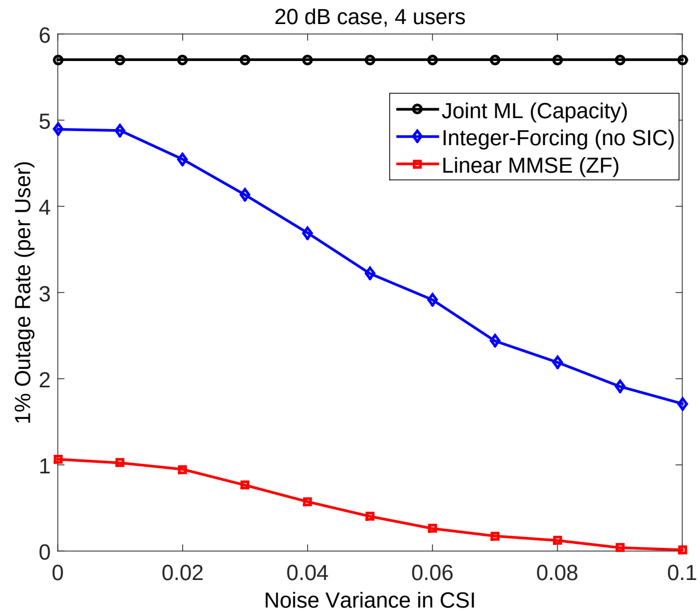

Figure 3·5: Robustness of IF to imperfect channel state information. Here we increase the noise for CSI estimation. The SNR is fixed for 20dB. We evalutate the performance using 1% outage rate and choose the average across 4 users.

channel

$$\tilde{\mathbf{y}}_{\mathrm{u},m}^{\mathsf{T}} = \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{Y}_{\mathrm{u}} - \sum_{\ell=1}^{m-1} r_{\mathrm{u},m,\ell}\, \mathbf{a}_{\mathrm{u},\ell}^{\mathsf{T}} \mathbf{S}_{\mathrm{u}} \tag{3.20}$$

$$= \mathbf{a}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{S}_{\mathrm{u}} + \mathbf{z}_{\mathrm{u,SIC},m}^{\mathsf{T}} \tag{3.21}$$

$$\mathbf{z}_{\mathrm{u,SIC},m}^{\mathsf{T}} \triangleq \left(\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{H}_{\mathrm{u}} \mathbf{C}_{\mathrm{u}} - \mathbf{r}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{A}_{\mathrm{u}}\right) \mathbf{S}_{\mathrm{u}} + \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{Z}_{\mathrm{u}} \ . \tag{3.22}$$

where $\mathbf{r}_{\mathrm{u},m}^{\mathsf{T}}$ is the m^{th} row of $\mathbf{R}_{\mathrm{u}}$ and $r_{\mathrm{u},m,\ell}$ is the $(m,\ell)^{\text{th}}$ entry. The *effective noise variance* is defined to be

$$\sigma_{\mathrm{u,SIC},m}^2 \triangleq \frac{1}{n} \mathbb{E} \|\mathbf{z}_{\mathrm{u,SIC},m}\|^2 \tag{3.23}$$

$$= \|\mathbf{b}_{\mathrm{u},m}\|^2 + \left\| \left(\mathbf{b}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{H}_{\mathrm{u}} \mathbf{C}_{\mathrm{u}} - \mathbf{r}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{A}_{\mathrm{u}}\right) \mathbf{P}_{\mathrm{u}}^{1/2} \right\|^2 \tag{3.24}$$

From [Nazer et al., 2016, Theorem 5], the following rates are achievable:

$$R_{\mathrm{u},m}^{\mathrm{SIC}} = \frac{1}{2}\log^{+}\left(\frac{P_{\mathrm{u},m}}{\sigma^2_{\mathrm{u,SIC},m}}\right) \quad m = 1, \ \ldots \ , L \ . \tag{3.25}$$

Here we rewrite Definition 2 for IF with SIC.

Definition 3. *We say that the identity permutation is admissible for a full-rank integer matrix* $\mathbf{A}_{\mathrm{u}}$ *if*

(a) the effective noise variances are in increasing order, $\sigma^2_{\mathrm{u,SIC},1} \leq \cdots \leq \sigma^2_{\mathrm{u,SIC},L}$ *and*

(b) there exists a lower unitriangular matrix $\mathbf{L} \in \mathbb{R}^{L\times L}$ *such that* $\mathbf{L}\mathbf{A}_{\mathrm{u}}$ *is upper triangular.*

Note that $\mathbf{R}_{\mathrm{u}}$ will also need to be modified, since it will not remain upper unitriangular under row permutation. Again, for the remainder of the paper, we assume that the rows and columns of $\mathbf{A}_{\mathrm{u}}$ have been reindexed so that these two conditions are satisfied.

Furthermore, for our decoding procedure, we will need to triangularize $\mathbf{A}_{\mathrm{u}}$ over $\mathbb{Z}_p$ in the following sense. We need a lower unit-triangular matrix $\bar{\mathbf{L}} \in \mathbb{Z}_p^{L\times L}$ such that $\bar{\mathbf{A}} = [\bar{\mathbf{L}}\mathbf{A}_{\mathrm{u}}] \bmod p$ is upper triangular. By [Ordentlich et al., 2014, Appendix A], such a matrix always exists if Definition 3 is satisfied. It also follows that $\bar{\mathbf{L}}$ has a lower unit-triangular inverse $\bar{\mathbf{L}}^{(\mathrm{inv})}$ over $\mathbb{Z}_p$.

Remark 1. *The uplink integer-forcing strategy without SIC is equivalent to setting* $\mathbf{R}_{\mathrm{u}} = \mathbf{I}$. *We omit a full description of this special case for the sake of brevity and state the achievable rates .*

Remark 2. *Although it is not immediately obvious, any rate tuple that is achievable via a conventional linear architecture is also achievable via an integer-forcing linear architecture by using the same beamforming matrix, setting the integer matrix to be the identity matrix, and scaling the equalization vectors by the appropriate MMSE coefficient [Zhan et al., 2014, Lemma 3]. While [Zhan et al., 2014] only establishes this for the uplink without SIC, this can be directly generalized to the SIC case as well as the downlink with or without DPC.*

3.2.3 Uplink Integer-Forcing Architecture with SIC

Recall from Chapter 2 that using the linear labeling φ, we can show that each nested lattice codebook $\mathcal{L}_\ell$ is isomorphic to the vector space $\mathbb{Z}_p^{k_{\mathrm{F},\ell}-k_{\mathrm{C},\ell}}$. Each user will take the p-ary expansion of its message index $w_{\mathrm{u},\ell}$ to obtain a message vector $\mathbf{w}_{\mathrm{u},\ell} \in \mathbb{Z}_p^{k_{\mathrm{F},\ell}-k_{\mathrm{C},\ell}}$. The intermediate goal of the receiver is to recover L linear combinations of the form

$$\mathbf{u}_{\mathrm{u},m} = \bigoplus_{\ell=1}^{L} q_{\mathrm{u},m,\ell} \tilde{\mathbf{w}}_{\mathrm{u},\ell} \tag{3.26}$$

where $q_{\mathrm{u},m,\ell} = [a_{\mathrm{u},m,\ell}] \bmod p$, $a_{\mathrm{u},m,\ell}$ is the $(m,\ell)^{\text{th}}$ entry of $\mathbf{A}_{\mathrm{u}}$, and $\tilde{\mathbf{w}}_{\mathrm{u},\ell} \in [\![\mathbf{w}_{\mathrm{u},\ell}]\!]$ with

$$[\![\mathbf{w}_{\mathrm{u},\ell}]\!] \triangleq \left\{ \mathbf{w} \in \mathbb{Z}_p^k : \mathbf{w} = \begin{bmatrix} \mathbf{e} \\ \mathbf{w}_{\mathrm{u},\ell} \\ \mathbf{0}_{k_\mathrm{F}-k_{\mathrm{F},\ell}} \end{bmatrix} \text{ for some } \mathbf{e} \in \mathbb{Z}_p^{k_{\mathrm{C},\ell}-k_\mathrm{C}} \right\}. \tag{3.27}$$

That is, the receiver attempts to recover L linear combinations of cosets of the messages. As discussed in [Nazer et al., 2016], the flexibility to choose $\mathbf{e}$ above seems to be necessary in order to permit unequal power allocation across the users via nested lattice codes.

We now state the encoding and decoding steps used in the successive integer-forcing architecture. We select an ensemble of good nested lattices

$$\Lambda_{\mathrm{C},1}, \ldots, \Lambda_{\mathrm{C},L}, \Lambda_{\mathrm{F},1}, \ldots, \Lambda_{\mathrm{F},L}$$

with parameters $P_{\mathrm{u},1}, \ldots, P_{\mathrm{u},L}$ and $\sigma^2_{\mathrm{SIC},1}, \ldots, \sigma^2_{\mathrm{SIC},L}$ using Theorem 1.

Encoding: The ℓ^{th} transmitter starts by taking the p-ary expansion of its message index $w_{\mathrm{u},\ell}$ to obtain the message vector $\mathbf{w}_{\mathrm{u},\ell} \in \mathbb{Z}_p^{k_{\mathrm{F},\ell}-k_{\mathrm{C},\ell}}$. It then uses the inverse linear labeling to map this to a lattice point

$$\boldsymbol{\lambda}_{\mathrm{u},\ell} = \left[\bar{\varphi}\left(\begin{bmatrix} \mathbf{0}_{k_{\mathrm{C},\ell}-k_\mathrm{C}} \\ \mathbf{w}_{\mathrm{u},\ell} \\ \mathbf{0}_{k_\mathrm{F}-k_{\mathrm{F},\ell}} \end{bmatrix} \right) \right] \bmod \Lambda_{\mathrm{C},\ell} \tag{3.28}$$

and dithers it to produce the codeword

$$\mathbf{s}_{\mathrm{u},\ell} = [\boldsymbol{\lambda}_{\mathrm{u},\ell} + \mathbf{d}_{\mathrm{u},\ell}] \bmod \Lambda_{\mathrm{C},\ell} \tag{3.29}$$

where the dither vector $\mathbf{d}_{\mathrm{u},\ell}$ is drawn independently and uniformly over $\mathcal{V}_{\mathrm{C},\ell}$. Thus, by the Crypto Lemma and Theorem 1(b), $\mathbf{s}_{\mathrm{u},\ell}$ is independent of $\boldsymbol{\lambda}_{\mathrm{u},\ell}$ and has power close to $P_{\mathrm{u},\ell}$. Finally, the transmitter uses its beamforming vector $\mathbf{c}_{\mathrm{u},\ell}$ to produce its channel input

$$\mathbf{X}_{\mathrm{u},\ell} = \mathbf{c}_{\mathrm{u},\ell}^{\mathsf{T}} \mathbf{s}_{\mathrm{u},\ell} \ . \tag{3.30}$$

Decoding: The receiver attempts to recover linear combinations of the form (3.26) one-by-one via successive cancellation and then solve them to obtain estimates of the message vectors. As an intermediate step, the receiver will attempt to decode certain integer-linear combinations of the lattice codewords, i.e.,

$$\boldsymbol{\mu}_{\mathrm{u},m} = \left[\sum_{\ell=1}^{L} a_{\mathrm{u},m,\ell} \, \tilde{\boldsymbol{\lambda}}_{\mathrm{u},\ell} \right] \bmod \Lambda_{\mathrm{C}} \tag{3.31}$$

where $\tilde{\boldsymbol{\lambda}}_{\mathrm{u},\ell} \triangleq \boldsymbol{\lambda}_{\mathrm{u},\ell} - Q_{\Lambda_{\mathrm{C},\ell}}(\boldsymbol{\lambda}_{\mathrm{u},\ell} + \mathbf{d}_{\mathrm{u},\ell})$. The linear labels of these integer-linear combinations correspond to the desired linear combinations, $\varphi(\boldsymbol{\mu}_{\mathrm{u},m}) = \mathbf{u}_{\mathrm{u},m}$. It will also attempt to recover integer-linear combinations of the dithered codewords, i.e.,

$$\mathbf{t}_{\mathrm{u},m} = \mathbf{a}_{\mathrm{u},m}^{\mathsf{T}} \mathbf{S}_{\mathrm{u}} \ . \tag{3.32}$$

The main obstacle is that, in order to decode the m^{th} integer-linear combination, the receiver must first cancel out the first $m-1$ codewords using the prior $m-1$ linear combinations. This is accomplished via the algebraic SIC technique from [Ordentlich et al., 2014]. Define

$$\boldsymbol{\nu}_{\mathrm{u},m} = \left[\boldsymbol{\mu}_{\mathrm{u},m} + \sum_{i=1}^{m-1} \bar{l}_{m,i}\boldsymbol{\mu}_{\mathrm{u},i}\right] \bmod \Lambda_{\mathrm{C}} \tag{3.33}$$

$$= \left[\sum_{\ell=1}^{L} \bar{a}_{m,\ell}\,\tilde{\boldsymbol{\lambda}}_{\mathrm{u},\ell}\right] \bmod \Lambda_{\mathrm{C}} \tag{3.34}$$

where $\bar{l}_{m,i}$ is the $(m,i)^{\text{th}}$ entry of $\bar{\mathbf{L}}$ and $\bar{a}_{m,\ell}$ is the $(m,\ell)^{\text{th}}$ entry of $\bar{\mathbf{A}}$ defined above. Note that $\boldsymbol{\nu}_{\mathrm{u},m} \in \Lambda_{\mathrm{F},m}$ and, given $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m$, we can recover $\boldsymbol{\mu}_{\mathrm{u},m}$:

$$\boldsymbol{\mu}_{\mathrm{u},m} = \left[\sum_{i=1}^{m} \bar{l}_{m,i}^{(\mathrm{inv})}\boldsymbol{\nu}_i\right] \bmod \Lambda_{\mathrm{C}} \tag{3.35}$$

where $\bar{l}_{m,i}^{(\mathrm{inv})}$ is the $(m,i)^{\text{th}}$ entry of $\bar{\mathbf{L}}^{(\mathrm{inv})}$.

We now proceed by induction. For the m^{th} decoding step, we assume that the receiver has already successfully recovered the previous $m-1$ integer-linear combinations, i.e., $\hat{\boldsymbol{\mu}}_{\mathrm{u},1} = \boldsymbol{\mu}_{\mathrm{u},1}, \ldots, \hat{\boldsymbol{\mu}}_{\mathrm{u},m-1} = \boldsymbol{\mu}_{\mathrm{u},m-1}$ and $\hat{\mathbf{t}}_{\mathrm{u},1} = \mathbf{t}_{\mathrm{u},1}, \ldots, \hat{\mathbf{t}}_{\mathrm{u},m-1} = \mathbf{t}_{\mathrm{u},m-1}$. The receiver uses this side information to form the effective channel output

$$\tilde{\mathbf{y}}_{\mathrm{u},m}^{\mathsf{T}} = \mathbf{b}_{\mathrm{u},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{u}} - \sum_{\ell=1}^{m-1} r_{\mathrm{u},m,\ell}\,\hat{\mathbf{t}}_{\mathrm{u},\ell}\ . \tag{3.36}$$

The receiver then removes the dithers[4], nulls out the lattice codewords corresponding to the first $m-1$ users, and quantizes onto the m^{th} fine lattice,

$$\boldsymbol{\nu}_{\mathrm{u},m} = \left[Q_{\Lambda_{\mathrm{F},m}}\left(\tilde{\mathbf{y}}_{\mathrm{u},m} + \sum_{i=1}^{m-1} \bar{l}_{m,i}\hat{\boldsymbol{\mu}}_{m,i} - \sum_{\ell=1}^{L} a_{\mathrm{u},m,\ell}\,\mathbf{d}_{\mathrm{u},\ell}\right)\right] \bmod \Lambda_{\mathrm{C}} \tag{3.37}$$

$$= \left[Q_{\Lambda_{\mathrm{F},m}}(\boldsymbol{\nu}_{\mathrm{u},m} + \mathbf{z}_{\mathrm{u,SIC},m})\right] \bmod \Lambda_{\mathrm{C}} \tag{3.38}$$

where the second step follows from [Nazer et al., 2016, §VI]. It then forms an estimate

[4]See [Nazer et al., 2016, Appendix H] for a proof that it suffices to use fixed dithers.

of its desired linear combination

$$\hat{\boldsymbol{\mu}}_{\mathrm{u},m} = \left[\sum_{i=1}^{m} \bar{l}_{m,i}^{(\mathrm{inv})} \hat{\boldsymbol{\nu}}_i\right] \bmod \Lambda_{\mathrm{C}} \tag{3.39}$$

$$\hat{\mathbf{u}}_{\mathrm{u},m} = \varphi(\hat{\boldsymbol{\mu}}_m) \ . \tag{3.40}$$

Finally, it attempts to recover the integer combination of the dithered codewords and proceed to the next round of induction,

$$\hat{\boldsymbol{\chi}}_m = \left[\hat{\boldsymbol{\mu}}_m + \sum_{\ell=1}^{L} a_{\mathrm{u},m,\ell}\, \mathbf{d}_{\mathrm{u},\ell}\right] \bmod \Lambda_{\mathrm{C}} \tag{3.41}$$

$$\hat{\mathbf{t}}_m = Q_{\Lambda_{\mathrm{C}}}(\tilde{\mathbf{y}}_m - \hat{\boldsymbol{\chi}}_m) + \hat{\boldsymbol{\chi}}_m \ . \tag{3.42}$$

After decoding all $\{\hat{\mathbf{u}}_{\mathrm{u},m} : m = 1, \ \cdots \ , L\}$, since $\mathbf{A}_{\mathrm{u}}$ is full rank, we can invert linear combinations

$$\begin{bmatrix} \hat{\mathbf{w}}_{\mathrm{u},1} \\ \vdots \\ \hat{\mathbf{w}}_{\mathrm{u},L} \end{bmatrix} = \mathbf{Q}_{\mathrm{u}}^{-1} \begin{bmatrix} \hat{\mathbf{u}}_{\mathrm{u},1} \\ \vdots \\ \hat{\mathbf{u}}_{\mathrm{u},L} \end{bmatrix} \ . \tag{3.43}$$

and solve for $\{\hat{\mathbf{w}}_{\mathrm{u},\ell} : \ell = 1, \ \cdots \ , L\}$. If for any $\epsilon > 0$ and n large enough, there exist encoders and decoder such that $\mathbb{P}\Big(\bigcup_{m=1}^{L}\{\hat{\mathbf{u}}_{\mathrm{u},m} \neq \mathbf{u}_{\mathrm{u},m}\}\Big) < \epsilon$, then $\mathbb{P}\Big(\bigcup_{m=1}^{L}\{\hat{\mathbf{w}}_{\mathrm{u},m} \neq \mathbf{w}_{\mathrm{u},m}\}\Big) < \epsilon$.

Theorem 2 ([Nazer et al., 2016, Lemma 13])**.** *For the successive integer-forcing architecture described above, the following rates are achievable*

$$R_{\mathrm{u},m}^{\mathrm{SIC}} = \frac{1}{2}\log^{+}\left(\frac{P_{\mathrm{u},m}}{\sigma_{\mathrm{u,SIC},m}^{2}}\right) \quad m = 1, \ \ldots \ , L \ . \tag{3.44}$$

For a full proof, see [Nazer et al., 2016, §VI].

Corollary 3. *For* $\mathbf{R}_{\mathrm{u}} = \mathbf{I}$*, the rates in* (3.19) *(corresponding to integer-forcing without SIC) are achievable.*

Chapter 4

IF in MIMO Broadcast Channel Coding (Downlink)

4.1 Downlink Architecture

4.1.1 Problem Statement

The downlink channel model mirrors the uplink channel model. There is a single N-antenna transmitter and L receivers. Let M_ℓ represent the number of antennas at the ℓ^{th} receiver and let $M = \sum_\ell M_\ell$ be the total number of receive antennas. The transmitter has L messages: the ℓ^{th} message $w_{\text{d},\ell}$ is drawn independently and uniformly from $\{1, 2, \ldots, 2^{nR_{\text{d},\ell}}\}$ and is intended for the ℓ^{th} receiver. The transmitter uses an *encoder* $\mathcal{E}_{\text{d}} : \{1, 2, \ldots, 2^{nR_{\text{d},1}}\} \times \{1, 2, \ldots, 2^{nR_{\text{d},L}}\} \to \mathbb{R}^{N \times n}$ to map these messages into a *channel input* $\mathbf{X}_{\text{d}} = \mathcal{E}_{\text{d}}(w_{\text{d},1}, \ldots, w_{\text{d},L})$ where n represents the blocklength. This channel input must satisfy a *total power constraint* $\mathbb{E}\Big[\operatorname{Tr}(\mathbf{X}_{\text{d}}\mathbf{X}_{\text{d}}^{\mathsf{T}})\Big] \leq nP_{\text{total}}$.

For $m = 1, \ldots, L$, the *channel output* observed by the m^{th} receiver is

$$\mathbf{Y}_{\text{d},m} = \mathbf{H}_{\text{d},m}\mathbf{X}_{\text{d}} + \mathbf{Z}_{\text{d},m} \tag{4.1}$$

where $\mathbf{H}_{\text{d},m} \in \mathbb{R}^{M_m \times N}$ is the channel matrix from the transmitter to the m^{th} receiver and the noise $\mathbf{Z}_{\text{d},m} \in \mathbb{R}^{M_m \times n}$ is elementwise i.i.d. Gaussian with mean zero and variance one. The receiver passes its channel output through a *decoder* $\mathcal{D}_{\text{d},m} : \mathbb{R}^{M_m \times n} \to \{1, 2, \ldots, 2^{nR_{\text{d},m}}\}$ in order to get an estimate $\hat{w}_{\text{d},m} = \mathcal{D}_{\text{d},m}(\mathbf{Y}_{\text{d},m})$ of its desired message.

Overall, we say that the downlink rates $R_{\mathrm{d},1}, \ldots, R_{\mathrm{d},L}$ are achievable if, for any $\epsilon > 0$ and n large enough, there exist an encoder and decoders such that $\mathbb{P}\Big(\bigcup_{\ell=1}^{L}\{\hat{w}_{\mathrm{d},\ell} \neq w_{\mathrm{d},\ell}\}\Big) < \epsilon$. The downlink capacity region is the closure of the set of all achievable rates.

Finally, it will often be useful to work with the following concatenated matrices,

$$\mathbf{Y}_{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{Y}_{\mathrm{d},1} \\ \vdots \\ \mathbf{Y}_{\mathrm{d},L} \end{bmatrix} \qquad \mathbf{H}_{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{H}_{\mathrm{d},1} \\ \vdots \\ \mathbf{H}_{\mathrm{d},L} \end{bmatrix} \qquad \mathbf{Z}_{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{Z}_{\mathrm{d},1} \\ \vdots \\ \mathbf{Z}_{\mathrm{d},L} \end{bmatrix} , \tag{4.2}$$

which enable us to compactly write the downlink channel output as

$$\mathbf{Y}_{\mathrm{d}} = \mathbf{H}_{\mathrm{d}}\mathbf{X}_{\mathrm{d}} + \mathbf{Z}_{\mathrm{d}} \ . \tag{4.3}$$

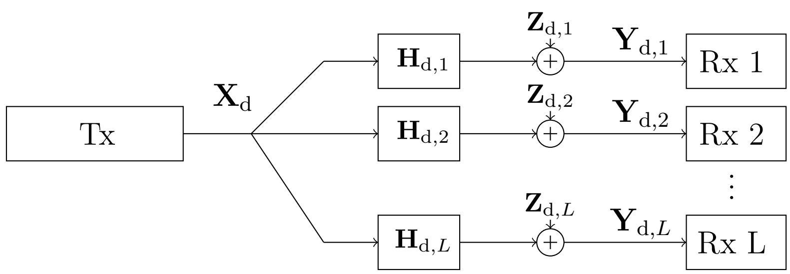

Figure 4·1: Block diagram of the downlink channel models.

4.1.2 Conventional Approach: ZF Linear Beamforming

The transmitter has a codeword $\mathbf{s}_{\mathrm{d},\ell} \in \mathbb{R}^n$ intended for the ℓ^{th} receiver with expected power $\frac{1}{n}\mathbb{E}\|\mathbf{s}_{\mathrm{d},\ell}\|^2 = P_{\mathrm{d},\ell}$. It collects these codewords into a matrix

$$\mathbf{S}_{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{s}_{\mathrm{d},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{s}_{\mathrm{d},L}^{\mathsf{T}} \end{bmatrix} \tag{4.4}$$

and applies a beamforming matrix $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N\times L}$ to create its channel input

$$\mathbf{X}_{\mathrm{d}} = \mathbf{B}_{\mathrm{d}}\mathbf{S}_{\mathrm{d}} \ . \tag{4.5}$$

The m^{th} receiver uses an equalization vector $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$ to form an effective channel output w

$$\mathbf{y}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{d}} \tag{4.6}$$

$$= \underbrace{\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}\mathbf{b}_{\mathrm{d},m}\mathbf{s}_{\mathrm{d},m}^{\mathsf{T}}}_{\text{signal}} + \underbrace{\sum_{\ell\neq m}\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},\ell}\mathbf{b}_{\mathrm{d},\ell}\mathbf{s}_{\mathrm{d},\ell}^{\mathsf{T}}}_{\text{interference}} + \underbrace{\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Z}_{\mathrm{d},m}}_{\text{noise}} \ . \tag{4.7}$$

Using i.i.d. Gaussian codewords, we can achieve the following rates:

$$R_{\mathrm{d},m} = \frac{1}{2}\log\left(1 + \frac{P_{\mathrm{d},m}\left|\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}\mathbf{b}_{\mathrm{d},m}\right|^2}{\sum_{\ell\neq m}P_{\mathrm{d},\ell}\left|\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},\ell}\mathbf{b}_{\mathrm{d},\ell}\right|^2}\right) \quad m = 1, \ \ldots \ , L \ . \tag{4.8}$$

4.1.3 Capacity Region for Downlink MIMO BC

As shown by [Weingarten et al., 2006], the downlink (i.e., MIMO BC) capacity region $\mathcal{C}_{\mathrm{d}}$ is the convex hull of the set of rate tuples $(R_{\mathrm{d},1}, \ldots, R_{\mathrm{d},L})$ satisfying

$$R_{\theta(\ell)} \leq \frac{1}{2}\log\left(\frac{\det\left(\mathbf{I} + \sum_{m\geq\ell}\mathbf{H}_{\mathrm{d},m}\mathbf{K}_m\mathbf{H}_{\mathrm{d},m}^{\mathsf{T}}\right)}{\det\left(\mathbf{I} + \sum_{m>\ell}\mathbf{H}_{\mathrm{d},m}\mathbf{K}_m\mathbf{H}_{\mathrm{d},m}^{\mathsf{T}}\right)}\right), \quad \ell = 1, \ \ldots \ , L \ . \tag{4.9}$$

for some permutation θ of $\{1, 2, \ldots, K\}$ and positive semi-definite matrices $\mathbf{K}_1, \ldots, \mathbf{K}_L$ satisfying the sum power constraint $\sum_{\ell=1}^{L}\mathrm{Tr}(\mathbf{K}_\ell) \leq P_{\text{total}}$. It can be attained using dirty-paper coding at the transmitter and joint typicality decoding at the receivers. See [Weingarten et al., 2006] or [El Gamal and Kim, 2011, §9.6.4] for more details.

Uplink-Downlink Duality. It can be argued that the uplink and downlink capacity regions described above are equal to one another, $\mathcal{C}_{\mathrm{u}} = \mathcal{C}_{\mathrm{d}}$. This was first shown for the sum-capacity [Vishwanath et al., 2003, Viswanath and Tse, 2003, Yu and Cioffi, 2004] and then for the full capacity region [Weingarten et al., 2006].

4.2 Downlink Integer-Forcing Architecture

The key idea underlying downlink integer-forcing is the fact that the transmitter can *pre-invert the linear combinations* prior to encoding. This technique, which was first proposed by Hong and Caire [Hong and Caire, 2012, Hong and Caire, 2013], allows each receiver to decode any integer-linear combination of the codewords in order to reduce the effective noise but still recover its desired message. These papers focused on the important special case where all users employ the same fine and coarse lattices, and thus have equal powers and must tolerate the worst effective noise across receivers. Below, we generalize their strategy to allow for unequal powers and a unique effective noise variance associated to each receiver. If the transmit antennas operate at different power levels, it is not possible to simply invert the linear combinations at the transmitter. Instead, for each symbol, we will need to apply the inverse of a submatrix that only includes the participating transmitters. Afterwards, we will introduce a dirty-paper integer-forcing scheme, building on the lattice-based dirty-paper strategy from [Zamir et al., 2002, Erez et al., 2005].

4.2.1 Downlink IF Architecture without DPC

We use the same encoding operations at the transmitter as in a conventional linear architecture. As in the uplink, we employ a nested lattice codebook to ensure that the codebook is closed under integer-linear combinations. As first proposed by Hong and Caire [Hong and Caire, 2012, Hong and Caire, 2013], we can also apply a precoding step over the finite field in order to “pre-invert” the linear combinations before mapping the messages to codewords. This step ensures that each receiver, upon recovering its integer-linear combination of codewords, can also obtain its desired message.

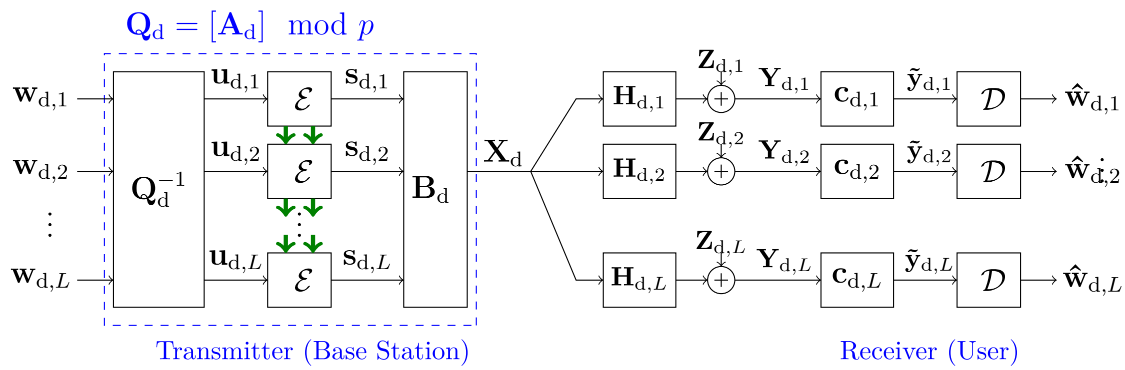

Figure 4·2: Block diagram of the integer-forcing downlink architecture. The encoder applies the inverse of $\mathbf{Q}_{\mathrm{d}} = [\mathbf{A}_{\mathrm{d}}] \mod p$ over $\mathbb{Z}_p$ to the message vectors $\mathbf{w}_{\mathrm{d},1}, \ldots, \mathbf{w}_{\mathrm{d},L}$ and then maps the results to dithered lattice codewords $\mathbf{s}_{\mathrm{d},1}, \ldots, \mathbf{s}_{\mathrm{d},L}$. The SISO encoders are possibly enhanced with dirty-paper coding (illustrated by green arrows). The channel input is formed by beamforming these codewords, $\mathbf{X}_{\mathrm{d}} = \mathbf{B}_{\mathrm{d}}\mathbf{S}_{\mathrm{d}}$. The m^{th} decoder uses an equalized channel output $\tilde{\mathbf{y}}_{\mathrm{d},m} = \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{d},m}$ to make an estimate of an integer-linear combination of the lattice codewords, which, due to the inverse operation corresponds to an estimate of its desired message.

The m^{th} receiver attempts to recover the linear combination $\mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{d}}$ where $\mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}$ is the m^{th} row of the full-rank, integer matrix $\mathbf{A}_{\mathrm{d}} \in \mathbb{Z}^{L\times L}$, i.e.,

$$\mathbf{A}_{\mathrm{d}} = \begin{bmatrix} \mathbf{a}_{\mathrm{d},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_{\mathrm{d},L}^{\mathsf{T}} \end{bmatrix} . \tag{4.10}$$

To do so, it uses an equalization vector $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$ to form the effective channel output

$$\tilde{\mathbf{y}}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{d},m} \tag{4.11}$$

$$= \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{d}} + \mathbf{z}_{\mathrm{d,eff},m}^{\mathsf{T}} \tag{4.12}$$

$$\mathbf{z}_{\mathrm{d,eff},m}^{\mathsf{T}} \triangleq \left(\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}\mathbf{B}_{\mathrm{d}} - \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\right)\mathbf{S}_{\mathrm{d}} + \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Z}_{\mathrm{d},m} . \tag{4.13}$$

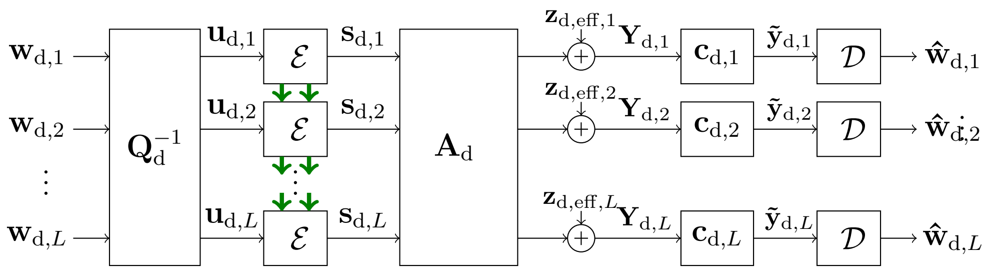

Figure 4·3: Block diagram of the effective channel induced by the integer-forcing downlink architecture. The m^{th} decoder observes an integer-linear combination of the codewords plus effective noise, $\sum_\ell a_{\mathrm{d},m,\ell}\mathbf{s}_{\mathrm{d},\ell} + \mathbf{z}_{\mathrm{d,eff},m}$. (If the encoders use dirty-paper coding, then $\mathbf{z}_{\mathrm{d,eff},m}$ is replaced with $\mathbf{z}_{\mathrm{u,DPC},m}$.) Since the encoder applied the inverse of $\mathbf{Q}_{\mathrm{d}} = [\mathbf{A}_{\mathrm{d}}] \bmod p$ over $\mathbb{Z}_p$ to the message vectors prior to mapping them to lattice codewords, then the m^{th} integer-linear combination corresponds to the m^{th} message.

We define the *effective noise variance* as

$$\sigma^2_{\mathrm{d,eff},m} \triangleq \frac{1}{n}\mathbb{E}\|\mathbf{z}_{\mathrm{d,eff},m}\|^2 \tag{4.14}$$

$$= \|\mathbf{c}_{\mathrm{d},m}\|^2 + \left\|\left(\mathbf{c}^{\mathsf{T}}_{\mathrm{d},m}\mathbf{H}_{\mathrm{d},m}\mathbf{B}_{\mathrm{d}} - \mathbf{a}^{\mathsf{T}}_{\mathrm{d},m}\right)\mathbf{P}_{\mathrm{d}}^{1/2}\right\|^2 . \tag{4.15}$$

As we will show in Theorem 4, the following rates are achievable

$$R_{\mathrm{d},m} = \frac{1}{2}\log^+\left(\frac{P_{\mathrm{d},m}}{\sigma^2_{\mathrm{d,eff},m}}\right) \quad m = 1, \ \ldots\ , L\ . \tag{4.16}$$

We begin by choosing a power allocation $\mathbf{P}_{\mathrm{d}} = \mathrm{diag}(P_{\mathrm{d},1}, \ldots, P_{\mathrm{d},L})$ for the codewords and a full-rank integer matrix $\mathbf{A}_{\mathrm{d}} \in \mathbb{Z}^{L\times L}$. We also select a beamforming matrix $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N\times L}$ and equalization vectors $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$, $m = 1, \ldots, L$. To meet the total power constraint with equality, we need that $\mathrm{Tr}(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}) \leq P_{\text{total}}$. Taken together, these choices specify the effective noise variances $\sigma^2_{\mathrm{d,eff},m}$ from (4.14).

As in the uplink case, the structure of the integer matrix $\mathbf{A}_{\mathrm{d}}$ will determine the order in which interference cancellation is possible via dirty-paper precoding. To simplify our notation, we will assume that $\mathbf{A}_{\mathrm{d}}$ is selected so that the m^{th} user can be associated with the m^{th} power $P_{\text{d},m}$. We specify when this is possible below.

Definition 4. *We say that the identity permutation is admissible for the downlink if*

(a) the powers are in decreasing order, $P_{\text{d},1} \geq \cdots \geq P_{\text{d},L}$ *and*

(b) the leading principal submatrices of $\mathbf{A}_{\text{d}}$ *are full rank,* $\text{rank}(\mathbf{A}_{\text{d}}^{[1:m]}) = m$ *for* $m = 1, \ldots, L$*.*

The first condition can be satisfied by reindexing the transmit antennas, which corresponds to reordering the powers, and permuting the columns of $\mathbf{A}_{\text{d}}$ and $\mathbf{B}_{\text{d}}$. The second condition can always be satisfied by reindexing the receivers, which corresponds to reordering the equalization vectors and permuting the rows of $\mathbf{A}_{\text{d}}$. To keep our notation manageable, we assume below that the rows and columns of $\mathbf{A}_{\text{d}}$ have been permuted so that Definition 4 holds.

We now describe the encoding and decoding steps used in the integer-forcing beamforming architecture. Using the parameters $P_{\text{d},1}, \ldots, P_{\text{d},L}$ and $\sigma^2_{\text{d,eff},1}, \ldots, \sigma^2_{\text{d,eff},L}$, we pick a good ensemble of nested lattices $\Lambda_{\text{C},1}, \ldots, \Lambda_{\text{C},L}, \Lambda_{\text{F},1}, \ldots, \Lambda_{\text{F},L}$ via Theorem 1. We will assume that the prime p used in the lattice construction is large enough so that $\mathbf{Q}_{\text{d}}^{[1:m]} = [\mathbf{A}_{\text{d}}^{[1:m]}] \bmod p$ is full rank over $\mathbb{Z}_p$ for $m = 1, \ldots, L$. It is always possible to choose such a prime, as argued in [Nazer et al., 2016, Lemmas 3, 4].

Encoding: Take the p-ary expansion of each message $w_{\text{d},\ell}$ to obtain the message vector $\mathbf{w}_\ell \in \mathbb{Z}_p^{k_{\text{F},\ell} - k_{\text{C},\ell}}$ for $\ell = 1, \ldots, L$. These vectors are then zero-padded to obtain

$$\bar{\mathbf{w}}_{\text{d},\ell} = \begin{bmatrix} \mathbf{0}_{k_{\text{C},\ell} - k_{\text{C}}} \\ \mathbf{w}_{\text{d},\ell} \\ \mathbf{0}_{k_{\text{F}} - k_{\text{F},\ell}} \end{bmatrix} . \tag{4.17}$$

We now proceed to pre-invert the linear combinations symbol-by-symbol. Recall that the notation $\mathbf{w}[i]$ refers to the i^{th} entry of the vector $\mathbf{w}$.

<u>Initialization Step, $k_{\text{C},L} - k_{\text{C}} + 1 \leq i \leq k$:</u> In this regime, all codewords have sufficient

power to participate, meaning that we can simply apply the inverse,

$$\begin{bmatrix} \mathbf{v}_{\mathrm{d},1}[i] \\ \vdots \\ \mathbf{v}_{\mathrm{d},L}[i] \end{bmatrix} = \mathbf{Q}_{\mathrm{d}}^{-1} \begin{bmatrix} \bar{\mathbf{w}}_{\mathrm{d},1}[i] \\ \vdots \\ \bar{\mathbf{w}}_{\mathrm{d},L}[i] \end{bmatrix} . \tag{4.18}$$

Note that the L^{th} codeword does not have sufficient power to control any other entries. Therefore, we set

$\mathbf{v}_{\mathrm{d},L}[1], \ldots, \mathbf{v}_{\mathrm{d},L}[k_{\mathrm{C},L} - k_{\mathrm{C}}] = 0$, apply the inverse linear labeling to obtain a fine lattice point

$$\boldsymbol{\lambda}_{\mathrm{d},L} = \bar{\varphi}(\mathbf{v}_{\mathrm{d},L}) , \tag{4.19}$$

and then generate our dithered codeword

$$\mathbf{s}_{\mathrm{d},L} = [\boldsymbol{\lambda}_{\mathrm{d},L} + \mathbf{d}_{\mathrm{d},L}] \bmod \Lambda_{\mathrm{C},L} \tag{4.20}$$

where the dither vector $\mathbf{d}_{\mathrm{d},L}$ is drawn independently and uniformly over $\mathcal{V}_{\mathrm{C},L}$. This codeword will contribute interference of the form

$$\mathbf{e}_{\mathrm{d},L} = \varphi\Big(Q_{\Lambda_{\mathrm{C},L}}(\boldsymbol{\lambda}_{\mathrm{d},L} + \mathbf{d}_{\mathrm{d},L})\Big) \tag{4.21}$$

to the remaining signal levels.

For the rest of the signal levels, we proceed by induction for $m = 1, \ldots, L-1$, assuming that $\mathbf{v}_{\mathrm{d},\ell}, \boldsymbol{\lambda}_{\mathrm{d},\ell}, \mathbf{s}_{\mathrm{d},\ell}, \mathbf{e}_{\mathrm{d},\ell}$ have been set for $\ell = m+1, \ldots, L$.

<u>Induction Step, $k_{\mathrm{C},m} - k_{\mathrm{C}} + 1 \leq i \leq k_{\mathrm{C},m+1}$</u>: In this regime, only the first m codewords have sufficient power to participate. Thus, we cancel out the interference caused by codewords $m+1, \ldots, L$ and apply the inverse of the m^{th} leading principal

submatrix,

$$\begin{bmatrix} \mathbf{v}_{\mathrm{d},1}[i] \\ \vdots \\ \mathbf{v}_{\mathrm{d},m}[i] \end{bmatrix} = \left(\mathbf{Q}_{\mathrm{d}}^{[1:m]}\right)^{-1} \begin{bmatrix} \bar{\mathbf{w}}_{\mathrm{d},1}[i] \oplus \bigoplus_{\ell=m+1}^{L} q_{\mathrm{d},1,\ell}\, \mathbf{e}_{\mathrm{d},\ell}[i] \\ \vdots \\ \bar{\mathbf{w}}_{\mathrm{d},m}[i] \oplus \bigoplus_{\ell=m+1}^{L} q_{\mathrm{d},m,\ell}\, \mathbf{e}_{\mathrm{d},\ell}[i] \end{bmatrix} . \tag{4.22}$$

Note that the m^{th} codeword does not have sufficient power to control any other entries. Therefore, we set $\mathbf{v}_{\mathrm{d},m}[1], \ldots, \mathbf{v}_{\mathrm{d},m}[k_{\mathrm{C},m} - k_{\mathrm{C}}] = 0$, apply the inverse linear labeling to obtain a fine lattice point

$$\boldsymbol{\lambda}_{\mathrm{d},m} = \bar{\varphi}(\mathbf{v}_{\mathrm{d},m}) \ , \tag{4.23}$$

and then generate our dithered codeword

$$\mathbf{s}_{\mathrm{d},m} = [\boldsymbol{\lambda}_{\mathrm{d},m} + \mathbf{d}_{\mathrm{d},m}] \bmod \Lambda_{\mathrm{C},m} \ . \tag{4.24}$$

where the dither vector $\mathbf{d}_{\mathrm{d},m}$[1] is drawn independently and uniformly over $\mathcal{V}_{\mathrm{C},m}$. This codeword will contribute digital interference of the form

$$\mathbf{e}_{\mathrm{d},m} = \varphi\Big(Q_{\Lambda_{\mathrm{C},m}}(\boldsymbol{\lambda}_{\mathrm{d},m} + \mathbf{d}_{\mathrm{d},m})\Big) \tag{4.25}$$

to the remaining signal levels.

After all signal levels have been set, we stack the dithered codewords

$$\mathbf{S}_{\mathrm{d}} = \begin{bmatrix} \mathbf{s}_{\mathrm{d},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{s}_{\mathrm{d},L}^{\mathsf{T}} \end{bmatrix} \tag{4.26}$$

and apply the beamforming matrix to create the channel input

$$\mathbf{X}_{\mathrm{d}} = \mathbf{B}_{\mathrm{d}} \mathbf{S}_{\mathrm{d}} \ . \tag{4.27}$$

[1] See [Nazer et al., 2016, Appendix H] for a proof that it suffices to use fixed dithers.

Decoding: The goal of each receiver is to decode its message vector $\mathbf{w}_{\mathrm{d},\ell}$. As a first step, it will make an estimate of the following integer-linear combination of the lattice codewords,

$$\boldsymbol{\mu}_{\mathrm{d},m} = \left[\sum_{\ell=1}^{L} a_{\mathrm{d},m,\ell}\,\tilde{\boldsymbol{\lambda}}_{\mathrm{d},\ell}\right] \bmod \Lambda_{\mathrm{C}} \tag{4.28}$$

where $a_{\mathrm{d},m,\ell}$ is the $(m,\ell)^{\text{th}}$ entry of $\mathbf{A}_{\mathrm{d}}$ and $\tilde{\boldsymbol{\lambda}}_{\mathrm{d},\ell} = \boldsymbol{\lambda}_{\ell} - Q_{\Lambda_{\mathrm{C},\ell}}(\boldsymbol{\lambda}_{\mathrm{d},\ell} + \mathbf{d}_{\mathrm{d},\ell})$. It forms its estimate by equalizing its observation

$$\tilde{\mathbf{y}}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{Y}_{\mathrm{d},m} \ , \tag{4.29}$$

removing the dither vectors, quantizing onto the m^{th} fine lattice, and taking the modulus with respect to the coarsest lattice,

$$\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \left[Q_{\Lambda_{\mathrm{F},m}}\left(\tilde{\mathbf{y}}_{\mathrm{d},m} - \sum_{\ell=1}^{L} a_{\mathrm{d},m,\ell}\,\mathbf{d}_{\mathrm{d},\ell}\right)\right] \bmod \Lambda_{\mathrm{C}} \ . \tag{4.30}$$

The linear label of this estimate can be viewed as an estimate of the desired message along with zero-padding,

$$\varphi(\hat{\boldsymbol{\mu}}_{\mathrm{d},m}) = \begin{bmatrix} \tilde{\mathbf{e}}_{\mathrm{d},m} \\ \hat{\mathbf{w}}_{\mathrm{d},m} \\ \mathbf{0}_{k_{\mathrm{F}}-k_{\mathrm{F},m}} \end{bmatrix} \ . \tag{4.31}$$

for some $\tilde{\mathbf{e}}_{\mathrm{d},m} \in \mathbb{Z}_p^{k_{\mathrm{C},m}-k_{\mathrm{C}}}$. As we will argue below, if $\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \boldsymbol{\mu}_{\mathrm{d},m}$, then $\hat{\mathbf{w}}_{\mathrm{d},m} = \mathbf{w}_{\mathrm{d},m}$.

Theorem 4. *Choose a power allocation* $\mathbf{P}_{\mathrm{d}} = \mathrm{diag}(P_{\mathrm{d},1}, \ldots, P_{\mathrm{d},L})$ *and full-rank integer matrix* $\mathbf{A}_{\mathrm{d}} \in \mathbb{Z}^{L\times L}$*. Assume, without loss of generality, that* $P_{\mathrm{d},1} \geq \cdots \geq P_{\mathrm{d},L}$*, and that* $\mathrm{rank}(\mathbf{A}^{[1:m]}) = m$ *for* $m = 1, \ldots, L$*. For a given beamforming matrix* $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N\times L}$*, channel matrices* $\mathbf{H}_{\mathrm{d},m} \in \mathbb{R}^{M_m\times N}$*, and equalization vectors* $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$*, the following rates are achievable*

$$R_{\mathrm{d},m} = \frac{1}{2}\log^{+}\left(\frac{P_{\mathrm{d},m}}{\sigma^2_{\mathrm{d,eff},m}}\right) \tag{4.32}$$

$$\sigma^2_{\mathrm{d,eff},m} = \|\mathbf{c}_{\mathrm{d},m}\|^2 + \left\|\left(\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}\mathbf{B}_{\mathrm{d}} - \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\right)\mathbf{P}_{\mathrm{d}}^{1/2}\right\|^2 \tag{4.33}$$

for $m = 1, \ldots, L$.

Proof. By the Crypto Lemma, each dithered codeword $\mathbf{s}_{\mathrm{d},\ell}$ is uniformly distributed over $\mathcal{V}_{\mathrm{C},\ell}$ and independent of the other dithered codewords. Thus, by Theorem 1(b), we have that $\frac{1}{n}\mathbb{E}\|\mathbf{s}_{\mathrm{d},\ell}\|^2 \leq P_{\mathrm{d},\ell}$, which guarantees that the power constraint is met

$$\frac{1}{n}\mathbb{E}\Big[\operatorname{Tr}(\mathbf{X}_{\mathrm{d}}^{\mathsf{T}}\mathbf{X}_{\mathrm{d}})\Big] = \frac{1}{n}\mathbb{E}\Big[\operatorname{Tr}(\mathbf{S}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{S}_{\mathrm{d}})\Big] \tag{4.34}$$

$$= \frac{1}{n}\mathbb{E}\Big[\operatorname{Tr}(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{S}_{\mathrm{d}}^{\mathsf{T}}\mathbf{S}_{\mathrm{d}})\Big] \tag{4.35}$$

$$= \frac{1}{n}\operatorname{Tr}\Big(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbb{E}[\mathbf{S}_{\mathrm{d}}^{\mathsf{T}}\mathbf{S}_{\mathrm{d}}]\Big) \tag{4.36}$$

$$\leq \frac{1}{n}\operatorname{Tr}\Big(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}\Big) = P_{\mathrm{total}}\ . \tag{4.37}$$

At the receiver side, we need to argue that $\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \boldsymbol{\mu}_{\mathrm{d},m}$ with high probability and, if so, $\hat{\mathbf{w}}_{\mathrm{d},m} = \mathbf{w}_{\mathrm{d},m}$. We begin by examining the linear labeling of $\boldsymbol{\mu}_{\mathrm{d},m}$,

$$\mathbf{u}_{\mathrm{d},m} = \varphi(\boldsymbol{\mu}_{\mathrm{d},m}) \tag{4.38}$$

$$= \bigoplus_{\ell=1}^{L} q_{\mathrm{d},m,\ell}\Big(\varphi(\boldsymbol{\lambda}_{\mathrm{d},\ell}) \ominus \varphi\Big(Q_{\Lambda_{\mathrm{C},\ell}}(\boldsymbol{\lambda}_{\mathrm{d},\ell} + \mathbf{d}_{\mathrm{d},\ell})\Big)\Big) \tag{4.39}$$

$$= \bigoplus_{\ell=1}^{L} q_{\mathrm{d},m,\ell}\Big(\varphi(\boldsymbol{\lambda}_{\mathrm{d},\ell}) \ominus \mathbf{e}_{\mathrm{d},\ell}\Big)\ . \tag{4.40}$$

Now, we examine the i^{th} symbol of this linear label for $k_{\mathrm{C},m} - k_{\mathrm{C}} + 1 \leq i \leq k$,

$$\mathbf{u}_{\mathrm{d},m}[i] = \bigoplus_{\ell=1}^{L} q_{\mathrm{d},m,\ell}\left(\mathbf{v}_{\mathrm{d},\ell}[i] \ominus \mathbf{e}_{\mathrm{d},\ell}[i]\right) \tag{4.41}$$

$$\overset{(a)}{=} \bigoplus_{\ell=1}^{m} q_{\mathrm{d},m,\ell}\,\mathbf{v}_{\mathrm{d},\ell}[i] \ominus \bigoplus_{\ell=m+1}^{L} q_{\mathrm{d},m,\ell}\,\mathbf{e}_{\mathrm{d},\ell}[i] \tag{4.42}$$

$$\overset{(b)}{=} \bar{\mathbf{w}}_{\mathrm{d},m}[i] \oplus \bigoplus_{\ell=1}^{m} q_{\mathrm{d},m,\ell}\,\mathbf{e}_{\mathrm{d},\ell}[i] \ominus \bigoplus_{\ell=m+1}^{L} q_{\mathrm{d},m,\ell}\,\mathbf{e}_{\mathrm{d},\ell}[i] \tag{4.43}$$

$$= \bar{\mathbf{w}}_{\mathrm{d},m}[i] \tag{4.44}$$

where (a) uses the fact that $\mathbf{v}_{\mathrm{d},\ell}[i] = 0$ for $\ell = m+1, \ldots, L$ by construction and $\mathbf{e}_{\mathrm{d},\ell}[i] = 0$ for $\ell = 1, \ldots, m$ via Definition 1(a) since $\mathbf{e}_{\mathrm{d},\ell}$ is the linear label of a lattice point from $\Lambda_{\mathrm{C},\ell}$ and (b) follows from plugging in (4.22). From (4.17) it follows that, if $\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \boldsymbol{\mu}_{\mathrm{d},m}$, then $\hat{\mathbf{w}}_{\mathrm{d},m} = \mathbf{w}_{\mathrm{d},m}$. Note that, since the last $k_{\mathrm{F}} - k_{\mathrm{F},m}$ entries of $\bar{\mathbf{w}}_{\mathrm{d},m}$

are zero, we know from Definition 1(a) that $\boldsymbol{\mu}_{\mathrm{d},m} \in \Lambda_{\mathrm{F},m}$.

We need to argue that $\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \boldsymbol{\mu}_{\mathrm{d},m}$ with probability at least $1 - \epsilon$. Recall from (4.12) and (4.13) that $\tilde{\mathbf{y}}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{d}} + \mathbf{z}_{\mathrm{d,eff},m}^{\mathsf{T}}$. Thus,

$$\tilde{\mathbf{y}}_{\mathrm{d},m} = \sum_{\ell=1}^{L} a_{\mathrm{d},m,\ell}\Big(\boldsymbol{\lambda}_{\mathrm{d}} + \mathbf{d}_{\mathrm{d},m} - Q_{\Lambda_{\mathrm{C},\ell}}(\boldsymbol{\lambda}_{\mathrm{d},\ell} + \mathbf{d}_{\mathrm{d},\ell})\Big) + \mathbf{z}_{\mathrm{d,eff},m} \tag{4.45}$$

$$= \sum_{\ell=1}^{L} a_{\mathrm{d},m,\ell}(\tilde{\boldsymbol{\lambda}}_{\mathrm{d},\ell} + \mathbf{d}_{\mathrm{d},\ell}) + \mathbf{z}_{\mathrm{d,eff},m} \ , \tag{4.46}$$

and, using (2.4),

$$\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \Big[Q_{\Lambda_{\mathrm{F},m}}(\boldsymbol{\mu}_{\mathrm{d},m} + \mathbf{z}_{\mathrm{d,eff},m})\Big] \bmod \Lambda_{\mathrm{C}} \ . \tag{4.47}$$

From Theorem 1(c), we know that, since $\boldsymbol{\mu}_{\mathrm{d},m} \in \Lambda_{\mathrm{F},m}$, the quantization step can tolerate noise with effective variance $\sigma^2_{\mathrm{d,eff},m}$, which implies that $\mathbb{P}(\hat{\boldsymbol{\mu}}_m \neq \boldsymbol{\mu}_m) < \epsilon$. Finally, from Theorem 1(d), we know that the rate satisfies $R_{\mathrm{d},m} > \frac{1}{2}\log^+(P_{\mathrm{d},m}/\sigma^2_{\mathrm{d,eff},m}) - \epsilon$. □

Remark 3. *If we do not wish to index the transmit antennas or receivers, the achievable rates can be expressed as follows. Let θ be a permutation that places the codeword powers in decreasing order. Also, let π be a permutation such that the leading principal submatrices of $\boldsymbol{\Theta}_\pi \mathbf{A}_{\mathrm{d}} \boldsymbol{\Theta}_\theta$ are full rank where $\boldsymbol{\Theta}_\pi$ and $\boldsymbol{\Theta}_\theta$ are the permutation matrices corresponding to π and θ, respectively. For a given beamforming matrix $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N \times L}$, channel matrices $\mathbf{H}_{\mathrm{d},m} \in \mathbb{R}^{M_m \times N}$, and equalization vectors $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$, the rates $R_{\mathrm{d},\pi(m)} = \frac{1}{2}\log^+\Big(P_{\mathrm{d},\theta(m)}/\sigma^2_{\mathrm{d,eff},\pi(m)}\Big)$, $m = 1, \ldots, L$ are achievable.*

4.2.2 Overview of Downlink Dirty-Paper Integer-Forcing

The integer-forcing transmitter described above carefully cancels out interference between receivers in the *digital domain*. Here, we argue that the performance can be further enhanced via dirty-paper coding in the analog domain. Prior work demonstrated that nested lattice codes are an ideal building block for dirty-paper strategies [Zamir et al., 2002, Erez et al., 2005], and serves as an inspiration for the scheme proposed below.

The encoding scheme is based on the nested lattice DPC technique from [Zamir et al., 2002, Erez et al., 2005], and will be discussed in detail in Section 4.2.3. Let $\mathbf{R}_{\mathrm{d}}$ be an upper unitriangular matrix. At a high level, the nested lattice codewords $\mathbf{s}_{\mathrm{d},1}, \ldots, \mathbf{s}_{\mathrm{d},L}$ are mapped into dirty-paper codewords $\mathbf{s}_{\mathrm{DPC},1}, \ldots, \mathbf{s}_{\mathrm{DPC},L}$ with the property that the m^{th} message can be recovered from $\mathbf{a}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{R}_{\mathrm{d}} \mathbf{S}_{\mathrm{DPC}}$ where

$$\mathbf{S}_{\mathrm{DPC}} \triangleq \begin{bmatrix} \mathbf{s}_{\mathrm{DPC},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{s}_{\mathrm{DPC},L}^{\mathsf{T}} \end{bmatrix} . \tag{4.48}$$

The dirty-paper codewords have the same expected power as the nested lattice codewords, and the encoder generates its channel input by applying the beamforming matrix to the dirty-paper codewords,

$$\mathbf{X}_{\mathrm{d}} = \mathbf{B}_{\mathrm{d}} \mathbf{S}_{\mathrm{DPC}} . \tag{4.49}$$

The m^{th} decoder uses its equalization vector to generate an effective channel output

$$\begin{aligned} \tilde{\mathbf{y}}_{\mathrm{d},m} &= \mathbf{c}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{Y}_{\mathrm{d},m} && (4.50) \\ &= \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{R}_{\mathrm{d}} \mathbf{S}_{\mathrm{DPC}} + \mathbf{z}_{\mathrm{d,DPC},m}^{\mathsf{T}} && (4.51) \end{aligned}$$

where

$$\mathbf{z}_{\mathrm{d,DPC},m}^{\mathsf{T}} \triangleq \left(\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{H}_{\mathrm{d},m} \mathbf{B}_{\mathrm{d}} - \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{R}_{\mathrm{d}} \right) \mathbf{S}_{\mathrm{DPC}} . \tag{4.52}$$

The *effective noise variance* is defined to be

$$\begin{aligned} \sigma_{\mathrm{d,DPC},m}^2 &\triangleq \frac{1}{n} \mathbb{E} \|\mathbf{z}_{\mathrm{d,DPC},m}\|^2 && (4.53) \\ &= \|\mathbf{c}_{\mathrm{d},m}\|^2 + \left\| \left(\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{H}_{\mathrm{d},m} \mathbf{B}_{\mathrm{d}} - \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}} \mathbf{R}_{\mathrm{d}} \right) \mathbf{P}_{\mathrm{d}}^{1/2} \right\|^2 . && (4.54) \end{aligned}$$

Overall, we will show in Theorem 5 that the following rates are achievable

$$R_{\mathrm{d},m}^{\mathrm{DPC}} = \frac{1}{2}\log^{+}\left(\frac{P_{\mathrm{d},m}}{\sigma^2_{\mathrm{d,DPC},m}}\right) \quad m = 1, \ \ldots \ , L \ . \tag{4.55}$$

4.2.3 Dirty-Paper Integer-Forcing Architecture

As before, we assume that the powers are in decreasing order and the leading principal submatrices of $\mathbf{A}_{\mathrm{d}}$ are full rank. We select a beamforming matrix $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N\times L}$ that meets the power constraint, $\mathrm{Tr}(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}) \leq P_{\mathrm{total}}$, as well as equalization vectors $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$, $m = 1,\ldots,L$. Finally, we choose a unitriangular matrix $\mathbf{R}_{\mathrm{d}}$ that specifies the coefficients used in the dirty-paper cancellation process. These choices determine the effective noise variances $\sigma^2_{\mathrm{d,DPC},m}$ from (4.52). Using the parameters $P_{\mathrm{d},1},\ldots,P_{\mathrm{d},L}$ and $\sigma^2_{\mathrm{d,DPC},1},\ldots,\sigma^2_{\mathrm{d,DPC},L}$, we pick a good ensemble of nested lattices $\Lambda_{\mathrm{C},1},\ldots,\Lambda_{\mathrm{C},L},\Lambda_{\mathrm{F},1},\ldots,\Lambda_{\mathrm{F},L}$ via Theorem 1. As before, we assume that the prime p is large enough so that $\mathbf{Q}_{\mathrm{d}}^{[1:m]} = [\mathbf{A}_{\mathrm{d}}^{[1:m]}] \bmod p$ is full rank over $\mathbb{Z}_p$ for $m = 1,\ldots,L$.

Encoding:

The encoding steps are identical from (4.17) to (4.21) for the initialization step. We also define the L^{th} dirty-paper codeword as

$$\mathbf{s}_{\mathrm{DPC},L} = \mathbf{s}_{\mathrm{d},L} \ . \tag{4.56}$$

For the rest of the signal levels, we proceed by induction for $m = 1,\ldots,L-1$, assuming that $\mathbf{v}_{\mathrm{d},\ell},\boldsymbol{\lambda}_{\mathrm{d},\ell},\mathbf{s}_{\mathrm{d},\ell},\mathbf{s}_{\mathrm{DPC},\ell},\mathbf{e}_{\mathrm{d},\ell}$ have been set for $\ell = m+1,\ldots,L$.

<u>Induction Step, $k_{\mathrm{C},m} - k_{\mathrm{C}} + 1 \leq i \leq k_{\mathrm{C},m+1}$</u>: The induction steps are the same as before from (4.22) to (4.24). We then map the dithered codeword to a dirty-paper codeword,

$$\mathbf{s}_{\mathrm{DPC},\ell} = \left[\mathbf{s}_{\mathrm{d},m} - \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\,\mathbf{s}_{\mathrm{DPC},\ell}\right] \bmod \Lambda_{\mathrm{C},m} \ . \tag{4.57}$$

This dirty-paper codeword will contribute digital interference of the form

$$\mathbf{e}_{\mathrm{d},m} = \varphi\left(Q_{\Lambda_{\mathrm{C},m}}\left(\boldsymbol{\lambda}_{\mathrm{d},m} + \mathbf{d}_{\mathrm{d},m} + \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\,\mathbf{s}_{\mathrm{DPC},\ell}\right)\right) \tag{4.58}$$

to the remaining signal levels.

After all signal levels have been set, we stack the dirty-paper codewords

$$\mathbf{S}_{\mathrm{DPC}} = \begin{bmatrix} \mathbf{s}_{\mathrm{DPC},1}^{\mathsf{T}} \\ \vdots \\ \mathbf{s}_{\mathrm{DPC},L}^{\mathsf{T}} \end{bmatrix} \tag{4.59}$$

and apply the beamforming matrix to create the channel input

$$\mathbf{X}_{\mathrm{d}} = \mathbf{B}_{\mathrm{d}}\mathbf{S}_{\mathrm{DPC}}\ . \tag{4.60}$$

Decoding: The decoding steps at each receiver are identical to those in (4.28) to (4.31) except that we define the lattice points in the integer-linear combination (4.28) by

$$\tilde{\boldsymbol{\lambda}}_{\mathrm{d},\ell} = \boldsymbol{\lambda}_{\ell} - Q_{\Lambda_{\mathrm{C},\ell}}\left(\boldsymbol{\lambda}_{\mathrm{d},\ell} + \mathbf{d}_{\mathrm{d},\ell} - \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\,\mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}}\right)\ . \tag{4.61}$$

We now establish the achievable rates for dirty-paper integer-forcing.

Theorem 5. *Choose a power allocation* $\mathbf{P}_{\mathrm{d}} = \mathrm{diag}(P_{\mathrm{d},1}, \ldots, P_{\mathrm{d},L})$ *and full-rank integer matrix* $\mathbf{A}_{\mathrm{d}} \in \mathbb{Z}^{L\times L}$*. Assume, without loss of generality, that* $P_{\mathrm{d},1} \geq \cdots \geq P_{\mathrm{d},L}$*, and that* $\mathrm{rank}(\mathbf{A}^{[1:m]}) = m$ *for* $m = 1, \ldots, L$*. For a given upper unitriangular dirty-paper matrix* $\mathbf{R}_{\mathrm{d}} \in \mathbb{R}^{L\times L}$*, beamforming matrix* $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{N\times L}$*, channel matrices* $\mathbf{H}_{\mathrm{d},m} \in \mathbb{R}^{M_m\times N}$*, and equalization vectors* $\mathbf{c}_{\mathrm{d},m} \in \mathbb{R}^{M_m}$*, the following rates are achievable*

$$R_{\mathrm{d},m}^{\mathrm{DPC}} = \frac{1}{2}\log^{+}\left(\frac{P_{\mathrm{d},m}}{\sigma_{\mathrm{d,DPC},m}^{2}}\right) \tag{4.62}$$

$$\sigma_{\mathrm{d,DPC},m}^{2} = \|\mathbf{c}_{\mathrm{d},m}\|^{2} + \left\|\left(\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}\mathbf{B}_{\mathrm{d}} - \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{R}_{\mathrm{d}}\right)\mathbf{P}_{\mathrm{d}}^{1/2}\right\|^{2}\ . \tag{4.63}$$

for $m = 1, \ldots, L$*.*

As part of the proof, we will need the following lemma.

Lemma 2. *Let $\mathbf{r}_{\mathrm{d},m}^{\mathsf{T}}$ be the m^{th} row of the upper unitriangular matrix $\mathbf{R}_{\mathrm{d}}$ used in the dirty-paper encoding process. We have that*

$$\mathbf{r}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{DPC}} = \tilde{\boldsymbol{\lambda}}_{\mathrm{d},m}^{\mathsf{T}} + \mathbf{d}_{\mathrm{d},m}^{\mathsf{T}} \ . \tag{4.64}$$

Proof.

$$\begin{aligned}
\mathbf{r}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{S}_{\mathrm{DPC}} &\overset{(a)}{=} \mathbf{s}_{\mathrm{DPC},m}^{\mathsf{T}} + \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}} \\
&= \left[\mathbf{s}_{\mathrm{d},m}^{\mathsf{T}} - \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}}\right] \bmod \Lambda_{\mathrm{C},m} + \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}} \\
&= \boldsymbol{\lambda}_{\mathrm{d},m}^{\mathsf{T}} + \mathbf{d}_{\mathrm{d},m}^{\mathsf{T}} - \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}} \\
&\quad - Q_{\Lambda_{\mathrm{C},m}}\left(\boldsymbol{\lambda}_{\mathrm{d},m}^{\mathsf{T}} + \mathbf{d}_{\mathrm{d},m}^{\mathsf{T}} - \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}}\right) + \sum_{\ell=m+1}^{L} r_{\mathrm{d},m,\ell}\, \mathbf{s}_{\mathrm{DPC},\ell}^{\mathsf{T}} \\
&\overset{(b)}{=} \tilde{\boldsymbol{\lambda}}_{\mathrm{d},m}^{\mathsf{T}} + \mathbf{d}_{\mathrm{d},m}^{\mathsf{T}}
\end{aligned}$$

where step (a) uses the fact that $\mathbf{R}_{\mathrm{d}}$ is upper unitriangular and step (b) uses (4.61). □

Proof of Theorem 5: We can show that the expected power constraints using the argument from the beginning of the proof of Theorem 4. We can also follow the steps in the proof of Theorem 4 to establish that $\mathbf{u}_{\mathrm{d},m}[i] = \bar{\mathbf{w}}_{\mathrm{d},m}[i]$ for $k_{\mathrm{C},m} - k_{\mathrm{C}} + 1 \leq i \leq k$ and that $\boldsymbol{\mu}_m \in \Lambda_{\mathrm{F},m}$. It remains to show that $\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \boldsymbol{\mu}_{\mathrm{d},m}$ with probability at least $1 - \epsilon$.

First, we can rewrite the m^{th} effective channel output as

$$\tilde{\mathbf{y}}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{R}_{\mathrm{d}}\mathbf{S}_{\mathrm{DPC}} + \mathbf{z}_{\mathrm{d,DPC},m}^{\mathsf{T}} \tag{4.65}$$

$$= \sum_{\ell=1}^{L} a_{\mathrm{d},m,\ell}\left(\tilde{\boldsymbol{\lambda}}_{\mathrm{d},m}^{\mathsf{T}} + \mathbf{d}_{\mathrm{d},m}^{\mathsf{T}}\right) + \mathbf{z}_{\mathrm{d,DPC},m}^{\mathsf{T}} \tag{4.66}$$

where $\mathbf{z}_{\mathrm{d,DPC},m}$ is defined in (4.52) and the last step uses Lemma 2 on each row of

$\mathbf{R}_\mathrm{d}\mathbf{S}_\mathrm{DPC}$. From (2.4), we have that

$$\hat{\boldsymbol{\mu}}_{\mathrm{d},m} = \left[Q_{\Lambda_{\mathrm{F},m}}(\boldsymbol{\mu}_{\mathrm{d},m} + \mathbf{z}_{\mathrm{d,DPC},m})\right] \bmod \Lambda_\mathrm{C}\ . \tag{4.67}$$

From Theorem 1(c), we know that, since $\boldsymbol{\mu}_{\mathrm{d},m} \in \Lambda_{\mathrm{F},m}$, the quantization step can tolerate noise with effective variance $\sigma^2_{\mathrm{d,DPC},m}$, which implies that $\mathbb{P}(\hat{\boldsymbol{\mu}}_m \neq \boldsymbol{\mu}_m) < \epsilon$. Finally, from Theorem 1(d), we know that the rate satisfies $R_{\mathrm{d},m} > \frac{1}{2}\log^+(P_{\mathrm{d},m}/\sigma^2_{\mathrm{d,DPC},m}) - \epsilon$. ■

Chapter 5

Integer-forcing Source Coding

5.1 Problem Statement

Consider a K-user distributed Gaussian source coding problem. For $k = 1, \dots, K$, the k^{th} user observes a length-n source $\mathbf{x}_k = [x_k[1] \ \cdots \ x_k[n]]^{\mathsf{T}}$. The K sources $(x_1[i], \dots, x_K[i])$ are generated i.i.d. for each $i = 1, \dots, n$ according to a joint Gaussian distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{K}_{xx}$. Let $\mathbf{X} \triangleq [\mathbf{x}_1 \ \cdots \ \mathbf{x}_K]^{\mathsf{T}}$ denote the matrix of source vectors.

The k^{th} user has an encoder $\mathcal{E}_k : \mathbb{R}^n \to \{1, \dots, 2^{nR_k^{\text{s}}}\}$ that maps its source into a message $w_k = \mathcal{E}_k(\mathbf{x}_k)$ of rate R_k^{s}. These messages are sent across bit pipes to the decoder, which then applies its decoding function $\mathcal{D} : \{1, \dots, 2^{nR_1^{\text{s}}}\} \times \cdots \times \{1, \dots, 2^{nR_K^{\text{s}}}\} \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to generate estimates $\left(\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_K\right) = \mathcal{D}(w_1, \dots, w_K)$.

A rate-distortion tuple $(R_1^{\text{s}}, \dots, R_K^{\text{s}}, D_1, \dots, D_K)$ is said to be achievable if there exists a sequence of encoders and decoders satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E} \|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2 \leq D_k, \quad k = 1, \dots, K \ . \tag{5.1}$$

Following [Ordentlich and Erez, 2013], we impose the additional requirement that the sources estimates are conditionally unbiased,

$$\mathbb{E}\Big[\mathbf{x}_k - \hat{\mathbf{x}}_k \Big| \mathbf{X}\Big] = \mathbf{0}, \quad k = 1, \dots, K \ . \tag{5.2}$$

As argued in [Ordentlich and Erez, 2013], although this is an unconventional requirement, it may emerge in certain scenarios, e.g., multiple relays that compress their observations for a central processor that wishes to treat the quantization noises as additive. Our future work will focus on establishing source-channel duality results for integer-forcing for the standard scenario where this requirement is not imposed.

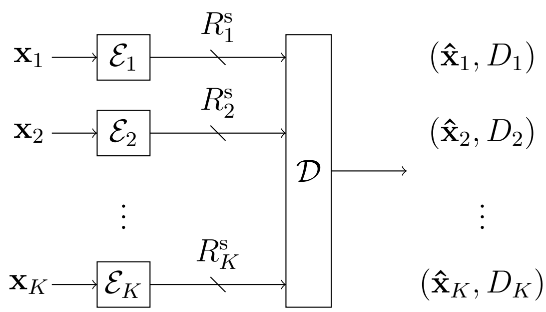

Figure 5·1: K-user distributed Gaussian source coding problem

5.2 Conventional Approach: Symmetric Rates and Distortions

We now summarize some of the nested lattice existence results from [Ordentlich and Erez, 2015b] which will be used in our coding scheme for IF source coding.

Lemma 3 ([Ordentlich et al., 2014, Lemma 3]). *Let $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_K$ be elements of the nested lattice chain from Theorem 1 and let $\mathbf{v}_1 = [\sum_k a_k \boldsymbol{\lambda}_k] \bmod \Lambda_m$ and $\mathbf{v}_2 = [\sum_k b_k \boldsymbol{\lambda}_k] \bmod \Lambda_m$ for $a_k, b_k \in \mathbb{Z}$. Then,*

$$[\mathbf{v}_1 + \mathbf{v}_2] \bmod \Lambda_m = \left[\sum_k ([a_k + b_k] \bmod p) \boldsymbol{\lambda}_k \right] \bmod \Lambda_m \ .$$

The basic integer-forcing source coding scheme with symmetric rate $R_1^{\mathrm{s}} = \ldots = R_K^{\mathrm{s}} = R_{\mathrm{sym}}^{\mathrm{s}}$ and distortion $D_1 = \ldots = D_K = D_{\mathrm{sym}}$ consists of the following steps.

Code Construction: For some choice of $\epsilon > 0$ and n large enough, select, using Theorem 1, a fine lattice Λ_{F} with parameter D_{sym} and a coarse lattice Λ_{C} with parameter θ_{C} to be specified later[1]. Generate K independent dither vectors $\mathbf{u}_1, \ldots, \mathbf{u}_K$ according to a uniform distribution over $\mathcal{V}_{\mathrm{F}}$.

Encoding: The k^{th} encoder dithers and quantizes its source onto Λ_{F} and then applies the modulo operation with respect to Λ_{C},

$$\boldsymbol{\lambda}_k = \Big[Q_{\Lambda_{\mathrm{F}}}(\mathbf{x}_k + \mathbf{u}_k)\Big] \bmod \Lambda_{\mathrm{C}} \ . \tag{5.3}$$

Decoding: Upon receiving the lattice codewords, the decoder removes the dithers to obtain

$$\begin{aligned}
\tilde{\mathbf{x}}_k &= [\boldsymbol{\lambda}_k - \mathbf{u}_k] \bmod \Lambda_{\mathrm{C}} && (5.4)\\
&= \Big[\mathbf{x}_k + Q_{\Lambda_{\mathrm{F}}}(\mathbf{x}_k + \mathbf{u}_k) - (\mathbf{x}_k + \mathbf{u}_k)\Big] \bmod \Lambda_{\mathrm{C}} && (5.5)\\
&= \Big[\mathbf{x}_k + \mathbf{z}_{\mathrm{eff},k}\Big] \bmod \Lambda_{\mathrm{C}} && (5.6)
\end{aligned}$$

where $\mathbf{z}_{\mathrm{eff},k} \triangleq -[\mathbf{x}_k + \mathbf{u}_k] \bmod \Lambda_{\mathrm{F}}$. Note that, by the Crypto Lemma, $\mathbf{z}_{\mathrm{eff},k}$ is uniform over $\mathcal{V}_{\mathrm{F}}$ and independent of $\mathbf{x}_k$. Define $\tilde{\mathbf{X}} = [\tilde{\mathbf{x}}_1 \ \cdots \ \tilde{\mathbf{x}}_K]^{\mathsf{T}}$ and $\mathbf{Z}_{\mathrm{eff}} = [\mathbf{z}_{\mathrm{eff},1} \ \cdots \ \mathbf{z}_{\mathrm{eff},K}]^{\mathsf{T}}$.

For some choice of full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{K \times K}$ with rows $\mathbf{a}_1^{\mathsf{T}}, \ldots, \mathbf{a}_K^{\mathsf{T}}$, the decoder forms K linear combinations

$$\begin{aligned}
\mathbf{v}_m &= [\mathbf{a}_m^{\mathsf{T}} \tilde{\mathbf{X}}] \bmod \Lambda_{\mathrm{C}} \qquad m = 1, \ldots, K && (5.7)\\
&= [\mathbf{a}_m^{\mathsf{T}} [\mathbf{X} + \mathbf{Z}_{\mathrm{eff}}] \bmod \Lambda_{\mathrm{C}}] \bmod \Lambda_{\mathrm{C}} && (5.8)\\
&= [\mathbf{a}_m^{\mathsf{T}} (\mathbf{X} + \mathbf{Z}_{\mathrm{eff}})] \bmod \Lambda_{\mathrm{C}} && (5.9)
\end{aligned}$$

where the last step uses the distributive law. By Theorem 1(b), if $\mathbf{a}_m^{\mathsf{T}}(\mathbf{K}_{xx} + D_{\mathrm{sym}}\mathbf{I})\mathbf{a}_m < \theta_{\mathrm{C}}$, then $\mathbf{v}_m = \mathbf{a}_m^{\mathsf{T}}(\mathbf{X} + \mathbf{Z}_{\mathrm{eff}})$ with probability at least $1 - \epsilon$. Thus, we set $\theta_{\mathrm{C}} = \max_m \mathbf{a}_m^{\mathsf{T}}(\mathbf{K}_{xx} + D_{\mathrm{sym}}\mathbf{I})\mathbf{a}_m + \epsilon$. Finally, the decoder inverts these linear combinations to obtain estimates of the form $\hat{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{z}_{\mathrm{eff}}$, each with distortion

[1] Here a distortion power D_ℓ can be considered as an effective noise power $\sigma^2_{\mathrm{eff},\ell}$ in Theorem 1 and θ_{C} can be considered as a symmetric coding power $P_1 = \ \cdots = \ P_L = P$ in Theorem 1.

at most D_{sym}. With probability at most $K\epsilon$, the squared norm of some $\mathbf{v}_m$ exceeds θ_C. The distortion in this case is still bounded and its contribution to the expected distortion vanishes as ϵ is driven to zero. Overall, the achievable rates are given by the following theorem.

Theorem 6 ([Ordentlich and Erez, 2013, Theorem 1])**.** *For a given full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{K\times K}$, the integer-forcing source coding strategy can achieve symmetric distortion D_{sym} with symmetric rate*

$$R_{sym} = \max_{m=1,\ldots,K} \frac{1}{2}\log\left(\mathbf{a}_m^{\mathsf{T}}(\mathbf{I} + D_{sym}^{-1}\mathbf{K}_{xx})\mathbf{a}_m\right) .$$

5.3 Coding Scheme for IF Source Coding

We now introduce successive integer-forcing. Our successive cancellation strategy relies on partially recovering m of the estimates after the m^{th} decoding step. Let $\mathbf{A}^{[\ell:k]}$ denote the submatrix of the matrix $\mathbf{A}$ consisting of only columns and rows numbered ℓ through k.

We begin by selecting a full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{K\times K}$ whose submatrices $\mathbf{A}^{[K:K]}, \mathbf{A}^{[K-1:K]}, \ldots, \mathbf{A}^{[2:K]}$ are also full rank. We can also state achievable rates for any full-rank integer matrix, if we introduce a permutation between the users and the coarse lattice parameters in (5.11). Throughout this chapter, we have strived to maintain the identity permutation for the sake of readability. We also select a unit, upper-triangular matrix $\mathbf{R}$ and write its m^{th} row as $\mathbf{r}_m^{\mathsf{T}}$ and its $(m,k)^{\text{th}}$ entry as $r_{m,k}$. Let $\mathbf{D} = \operatorname{diag}(D_1,\ldots,D_K)$ be the diagonal matrix of distortions.

Code Construction: For some choice of $\epsilon > 0$ and n large enough, select, using Theorem 1, an ensemble of fine lattices $\Lambda_{\mathrm{F},1},\ldots,\Lambda_{\mathrm{F},K}$ and coarse lattices $\Lambda_{\mathrm{C},1},\ldots,\Lambda_{\mathrm{C},K}$

according to parameters

$$\theta_{\mathrm{F},m} = D_m \tag{5.10}$$

$$\theta_{\mathrm{C},m} = \mathbf{r}_m^{\mathsf{T}} \mathbf{A}(\mathbf{K}_{xx} + \mathbf{D})\mathbf{A}^{\mathsf{T}} \mathbf{r}_m + \epsilon \tag{5.11}$$

for $m = 1, \ldots, K$. We assume that $\mathbf{R}$ and $\mathbf{A}$ are chosen such that

$$\theta_{\mathrm{C},1} \geq \cdots \geq \theta_{\mathrm{C},K} \quad , \tag{5.12}$$

which in turn ensures that $\Lambda_{\mathrm{C},1} \subseteq \cdots \subseteq \Lambda_{\mathrm{C},K}$. For $k = 1, \ldots, K$, generate an independent dither vector $\mathbf{u}_k$ according to a uniform distribution over $\mathcal{V}_{\mathrm{F},k}$.

Encoding: As before, the k^{th} encoder dithers and quantizes its source onto $\Lambda_{\mathrm{F},k}$ and then applies the modulo operation with respect to $\Lambda_{\mathrm{C},k}$,

$$\boldsymbol{\lambda}_k = \left[Q_{\Lambda_{\mathrm{F},k}}(\mathbf{x}_k + \mathbf{u}_k)\right] \bmod \Lambda_{\mathrm{C},k} \ .$$

Decoding: As in the symmetric case, the decoder begins by removing the dithers from its received lattice points,

$$\tilde{\mathbf{x}}_k = [\boldsymbol{\lambda}_k - \mathbf{u}_k] \bmod \Lambda_{\mathrm{C},k} \tag{5.13}$$

$$= \left[\mathbf{x}_k + \mathbf{z}_{\mathrm{eff},k}\right] \bmod \Lambda_{\mathrm{C}}, k \tag{5.14}$$

where $\mathbf{z}_{\mathrm{eff},k} \triangleq -[\mathbf{x}_k + \mathbf{u}_k] \bmod \Lambda_{\mathrm{F},k}$ and, by the Crypto Lemma, $\mathbf{z}_{\mathrm{eff},k}$ is uniform over $\mathcal{V}_{\mathrm{F},k}$ and independent of $\mathbf{x}_k$. As before, define $\tilde{\mathbf{X}} = [\tilde{\mathbf{x}}_1 \ \cdots \ \tilde{\mathbf{x}}_K]^{\mathsf{T}}$ and $\mathbf{Z}_{\mathrm{eff}} = [\mathbf{z}_{\mathrm{eff},1} \ \cdots \ \mathbf{z}_{\mathrm{eff},K}]^{\mathsf{T}}$.

We first recover the K^{th} linear combination,

$$\mathbf{v}_K = \left[\sum_k a_{K,k} \tilde{\mathbf{x}}_k\right] \bmod \Lambda_{\mathrm{C},K} \tag{5.15}$$

$$= \left[\sum_k a_{K,k}(\mathbf{x}_k + \mathbf{z}_{\mathrm{eff},k})\right] \bmod \Lambda_{\mathrm{C},K} \tag{5.16}$$

where the last step uses the distributive law. Using Theorem 1(b) and the fact that

$$\theta_{\mathrm{C},K} > \mathbf{r}_K^{\mathsf{T}}\mathbf{A}(\mathbf{K}_{xx}+\mathbf{D})\mathbf{A}^{\mathsf{T}}\mathbf{r}_K \tag{5.17}$$

$$= \mathbf{a}_K^{\mathsf{T}}(\mathbf{K}_{xx}+\mathbf{D})\mathbf{a}_K \tag{5.18}$$

we have that $\mathbf{v}_K = \mathbf{a}_K^{\mathsf{T}}(\mathbf{X}+\mathbf{Z}_{\text{eff}})$ with probability at least $1-\epsilon$.

We proceed by induction. Assume that $\mathbf{v}_\ell = \mathbf{a}_\ell^{\mathsf{T}}(\mathbf{X}+\mathbf{Z}_{\text{eff}})$ has been successfully recovered for $\ell = m{+}1,\ldots,K$. We now try to recover the m^{th} such linear combination. First, we need the following lemma.

Lemma 4. *Given $\mathbf{v}_\ell = \mathbf{a}_\ell^{\mathsf{T}}(\mathbf{X}+\mathbf{Z}_{\text{eff}})$ for $\ell = m+1,\ldots,K$ and $\tilde{\mathbf{x}}_k$ for $k=1,\ldots,K$, we can recover*

$$\mathbf{t}_{m,k} \triangleq [\mathbf{x}_k + \mathbf{z}_{\text{eff},k}] \bmod \Lambda_{\mathrm{C},m} \quad k=1,\ldots,K \ .$$

Proof of lemma 4. For $k=1,\ldots,m$, since $\Lambda_{\mathrm{C},k} \subseteq \Lambda_{\mathrm{C},m}$, we can directly calculate $[\tilde{\mathbf{x}}_k] \bmod \Lambda_{\mathrm{C},m} = [\mathbf{x}_k{+}\mathbf{z}_{\text{eff},k}] \bmod \Lambda_{\mathrm{C},m}$. For $k = m{+}1,\ldots,K$, we have $K{-}m$ modulo-lattice equations in K variables, m of which are known from above. Therefore, we can calculate

$$\mathbf{w}_k = \left[\mathbf{v}_k - \sum_{\ell=1}^{m} a_{k,\ell}\mathbf{t}_{m,\ell}\right] \bmod \Lambda_{\mathrm{C},m} \tag{5.19}$$

$$= \left[\sum_{\ell=m+1}^{K} a_{k,\ell}(\mathbf{x}_\ell + \mathbf{z}_{\text{eff},\ell})\right] \bmod \Lambda_{\mathrm{C},m} \ . \tag{5.20}$$

for $k = m+1,\ldots,K$. By assumption, the submatrix $\mathbf{A}^{[m:K]}$ is full rank. From [Ordentlich et al., 2014, Appendix A], it can be shown that for prime p large enough, $[\mathbf{A}^{[m:K]}] \bmod p$ is full rank over $\mathbb{Z}/p\mathbb{Z}$ as well. Thus, using Lemma 3, we can solve for the remaining terms by applying the inverse of $[\mathbf{A}^{[m:K]}] \bmod p$ over $\mathbb{Z}/p\mathbb{Z}$ to $[\mathbf{w}_{m+1} \cdots \mathbf{w}_K]^{\mathsf{T}}$. □

Combining the $\mathbf{t}_{m,k}$ from Lemma 4 with $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_K$, we obtain

$$\begin{aligned}\mathbf{s}_m &= \left[\sum_{k=1}^{K} a_{m,k}\mathbf{t}_{m,k} + \sum_{k=m+1}^{K} r_{m,k}\mathbf{v}_k\right] \bmod \Lambda_{\mathrm{C},m} \qquad (5.21)\\ &= \left[\sum_{k=1}^{K} a_{m,k}(\mathbf{x}_k + \mathbf{z}_{\mathrm{eff},k}) + \sum_{k<m} r_{m,k}\mathbf{v}_k\right] \bmod \Lambda_{\mathrm{C},m}\\ &= [\mathbf{r}_m^{\mathsf{T}}\mathbf{A}(\mathbf{X} + \mathbf{Z}_{\mathrm{eff}})] \bmod \Lambda_{\mathrm{C},m} \ . \qquad (5.22)\end{aligned}$$

Combining Theorem 1 with the fact that

$$\theta_{\mathrm{C},m} > \mathbf{r}_m^{\mathsf{T}}\mathbf{A}(\mathbf{K}_{xx} + \mathbf{D})\mathbf{A}^{\mathsf{T}}\mathbf{r}_m \ , \tag{5.23}$$

we find that, with probability at least $1 - \epsilon$,

$$\mathbf{s}_m = \mathbf{r}_m^{\mathsf{T}}\mathbf{A}(\mathbf{X} + \mathbf{Z}_{\mathrm{eff}}) \ . \tag{5.24}$$

Now, we remove the successive cancellation terms to recover

$$\mathbf{v}_m = \mathbf{s}_m - \sum_{k=m+1}^{K} r_{m,k}\mathbf{v}_k = \mathbf{a}_m^{\mathsf{T}}(\mathbf{X} + \mathbf{Z}_{\mathrm{eff}}) \tag{5.25}$$

with probability of error at most ϵ, which completes the induction step.

Finally, we have obtained $\mathbf{A}(\mathbf{X} + \mathbf{Z}_{\mathrm{eff}})$ with probability of error at most $K\epsilon$. Inverting $\mathbf{A}$, we obtain our estimates $\hat{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{z}_{\mathrm{eff},k}$, each with distortion D_k. In the event of a decoding failure, the distortion remains bounded and thus has a vanishing effect on the overall expected distortion as ϵ is taken to zero.

Theorem 7. *For any full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{K\times K}$ with full-rank submatrices $\mathbf{A}^{[K:K]}, \ldots, \mathbf{A}^{[2:K]}$ and unit upper-triangular matrix $\mathbf{R}$ satisfying (5.12), the following rates are achievable via successive integer-forcing source coding*

$$R_k^s = \frac{1}{2}\log\left(\frac{\mathbf{r}_k^{\mathsf{T}}\mathbf{A}(\mathbf{K}_{xx} + \mathbf{D})\mathbf{A}^{\mathsf{T}}\mathbf{r}_k}{D_k}\right) \quad k = 1, \ldots, K. \tag{5.26}$$

Chapter 6

Duality

6.1 Uplink-Downlink Duality for IF

6.1.1 Uplink-Downlink Duality

In previous sections, we show the achievable computation rate tuple in the uplink and downlink channel. We show successive integer-forcing technique improves the performance of integer-forcing in the uplink. We also show dirty-paper coding for compute-and-forward improves the performance of integer-forcing in the downlink channel. In this section, we demonstrate a duality connection between integer-forcing for the MIMO MAC and the MIMO BC. The duality connection allows us to establish an equality for the sum rate between the uplink and downlink channel with the same total power usage. We also generalize this result to include successive integer-forcing and dirty-paper integer-forcing.

For simplicity, we re-index rows of integer matrices $\mathbf{A}_{\mathrm{u}}$ and $\mathbf{A}_{\mathrm{d}}$ (re-index users) in the uplink and downlink channel such that the valid permutation orders from Definition 2 and Remark 3 are identity matrices. For the case without re-indexing, the proof of duality is similar, see [He et al., 2014] for details.

Recall that

$$\mathbf{C}_{\mathrm{u}} \triangleq \begin{bmatrix} \mathbf{c}_{\mathrm{u},1} & \mathbf{0}_{M_1} & \dots & \mathbf{0}_{M_1} \\ \mathbf{0}_{M_2} & \mathbf{c}_{\mathrm{u},2} & \dots & \mathbf{0}_{M_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{M_L} & \mathbf{0}_{M_L} & \dots & \mathbf{c}_{\mathrm{u},L} \end{bmatrix} \tag{6.1}$$

is the beamforming matrix in the uplink MAC and

$$\mathbf{C}_{\mathrm{d}} \triangleq \begin{bmatrix} \mathbf{c}_{\mathrm{d},1}^{\mathsf{T}} & \mathbf{0}_{M_1}^{\mathsf{T}} & \cdots & \mathbf{0}_{M_1}^{\mathsf{T}} \\ \mathbf{0}_{M_2}^{\mathsf{T}} & \mathbf{c}_{\mathrm{d},2}^{\mathsf{T}} & \cdots & \mathbf{0}_{M_2}^{\mathsf{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{M_L}^{\mathsf{T}} & \mathbf{0}_{M_L}^{\mathsf{T}} & \cdots & \mathbf{c}_{\mathrm{d},L}^{\mathsf{T}} \end{bmatrix} \tag{6.2}$$

is the equalization(projection) matrix in the downlink BC, $\mathbf{B}_{\mathrm{u}}$ is the uplink equalization matrix and $\mathbf{B}_{\mathrm{d}}$ is the downlink beamforming matrix. Here $\mathbf{P}_{\mathrm{u}}$ and $\mathbf{P}_{\mathrm{d}}$ are diagonal coding power matrices for the uplink and downlink, respectively. We will define $\boldsymbol{\rho}_{\mathrm{c,u}} = [P_{\mathrm{u},1}, \ \cdots \ , P_{\mathrm{u},L}]^{\mathsf{T}}$ and $\boldsymbol{\rho}_{\mathrm{c,d}} = [P_{\mathrm{d},1}, \ \cdots \ , P_{\mathrm{d},L}]^{\mathsf{T}}$ be the coding power vectors for the uplink and downlink. Here $\mathbf{R}_{\mathrm{u}}$ and $\mathbf{R}_{\mathrm{d}}$ are both unit triangular matrix which will only be used when we apply SIC in the uplink (if not, $\mathbf{R}_{\mathrm{u}} = \mathbf{I}$) and DPC in the downlink (if not, $\mathbf{R}_{\mathrm{d}} = \mathbf{I}$).

Theorem 8. *For a given uplink channel matrix $\mathbf{H}_{\mathrm{u}}$, integer matrix $\mathbf{A}_{\mathrm{u}}$, and (diagonal) power matrix $\mathbf{P}_{\mathrm{u}}$ that meets the total power constraint $\mathrm{Tr}(\mathbf{C}_{\mathrm{u}}^{\mathsf{T}}\mathbf{C}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}) = P_{\mathrm{total}}$, let $R_{\mathrm{u},1}, \ldots, R_{\mathrm{u},L}$ be a computation rate tuple that is achievable with projection matrix $\mathbf{B}_{\mathrm{u}}$ and precoding matrix $\mathbf{C}_{\mathrm{u}}$. Then, for the downlink channel matrix $\mathbf{H}_{\mathrm{d}} = \mathbf{H}_{\mathrm{u}}^{\mathsf{T}}$, integer matrix $\mathbf{A}_{\mathrm{d}} = \mathbf{A}_{\mathrm{u}}^{\mathsf{T}}$, there exists a unique (diagonal) power matrix $\mathbf{P}_{\mathrm{d}}$ with total power usage $\mathrm{Tr}(\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}) = P_{\mathrm{total}}$, such that the sum computation rate $\sum_i R_{\mathrm{d},i} = \sum_i R_{\mathrm{u},i}$ is achievable using (diagonal) projection matrix $\mathbf{C}_{\mathrm{d}} = \mathbf{C}_{\mathrm{u}}^{\mathsf{T}}$ and precoding matrix $\mathbf{B}_{\mathrm{d}} = \mathbf{B}_{\mathrm{u}}^{\mathsf{T}}$. The same relationship can be established starting from an achievable rate tuple for the downlink and going to the uplink.*

Theorem 9. *The duality result in Theorem 8 still holds for the uplink channel with successive integer-forcing and the downlink channel with DPC for integer-forcing, as long as the coefficients matrices for SIC and DPC satisfy $\mathbf{R}_{\mathrm{d}} = \mathbf{R}_{\mathrm{u}}^{\mathsf{T}}$.*

Remark 4. *We simplify our proof by assuming identity is a valid permutation order in both the uplink IF and downlink IF. We can make this assumption true by re-indexing users and rows of integer matrices ($\mathbf{A}_{\mathrm{u}}$ and $\mathbf{A}_{\mathrm{d}}$). However, we can not maintain this assumption for both the uplink IF and downlink IF simultaneously if $\mathbf{A}_{\mathrm{u}} = \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}$. The re-index process in the uplink and downlink might conflict with each other. For the following proof, we will first make a virtual assumption such that when $\mathbf{A}_{\mathrm{u}} = \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}$ there is a way to do re-indexing such that identity is a valid permutation in* *both uplink channel and downlink channel. This assumption will allow us to establish a duality that both the uplink IF and downlink IF can achieve the same rate tuple. We then back off and removing the virtual assumption. After removing the assumption, the same rate tuple result no longer holds, but we can still achieve the same sum computation rate, thus we have the results in Theorem 8 and Theorem 9.*

For the rest of this section, we will prove Theorem 9 only since Theorem 8 is a special case in Theorem 9 by setting $\mathbf{R}_{\mathrm{d}}$ and $\mathbf{R}_{\mathrm{u}}$ to be identity. Our proof is inspired by the approach of [Viswanath and Tse, 2003]. Before we start the proof, we need some basic results for non-negative matrices:

1. A vector or a matrix is *non-negative* (i.e., $\mathbf{F} \geq 0$) if all its entries are non-negative.

2. A vector or a matrix is *positive* (i.e., $\mathbf{F} > 0$) if all its entries are positive.

3. A square matrix $\mathbf{F}$ is a *Z-matrix* if all its off-diagonal elements are non-positive.

4. An *M-matrix* is a Z-matrix with eigenvalues whose real parts are positive.

Lemma 5. *Let $\mathbf{F}$ be a square Z-matrix. The following statements are equivalent:*

- *i) $\mathbf{F}$ is a non-singular M-matrix with a non-negative inverse ($\mathbf{F}^{-1}$ exists and $\mathbf{F}^{-1} \geq 0$).*

- *ii) There exists $\mathbf{x} \geq 0$ satisfying $\mathbf{F}\mathbf{x} > 0$.*

- *iii) Every real eigenvalue of $\mathbf{F}$ is positive.*

See [Plemmons, 1977] for a proof.

Proof of Theorem 9. Let $\gamma_\ell \triangleq P_{\mathrm{u},\ell}/\sigma^2_{\mathrm{u,SIC},\ell}$ denote the SINR for the ℓ^{th} uplink user in (3.44) and $\beta_\ell \triangleq P_{\mathrm{d},\ell}/\sigma^2_{\mathrm{d,DPC},\ell}$ denote the SINR for the ℓ^{th} downlink user in (4.62). Define $\boldsymbol{\gamma} \triangleq [\gamma_1 \ \cdots \ \gamma_L]^\mathsf{T}$ and $\boldsymbol{\beta} \triangleq [\beta_1 \ \cdots \ \beta_L]^\mathsf{T}$. Recall that $\boldsymbol{\rho}_{\mathrm{c,u}} = [P_{\mathrm{u},1}, \ \cdots \ , P_{\mathrm{u},L}]^\mathsf{T}$ and $\boldsymbol{\rho}_{\mathrm{c,d}} = [P_{\mathrm{d},1}, \ \cdots \ , P_{\mathrm{d},L}]^\mathsf{T}$ are the coding power vectors for the uplink and downlink, respectively.

Define $\mathbf{J}_{\mathrm{u}} \triangleq \operatorname{diag}\Big(\big[\|\mathbf{b}_{\mathrm{u},1}^{\mathsf{T}}\|^2 \cdots \|\mathbf{b}_{\mathrm{u},L}^{\mathsf{T}}\|^2\big]\Big)$, and $\mathbf{M}_{\mathrm{u}} \in \mathbb{R}^{L\times L}$ where $[\mathbf{M}_{\mathrm{u}}]_{ij} = (\mathbf{b}_{\mathrm{u},i}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}\mathbf{c}_{\mathrm{u},j} - \mathbf{r}_{\mathrm{u},i}^{\mathsf{T}}\hat{\mathbf{a}}_{\mathrm{u},j})^2$. Here $\mathbf{c}_{\mathrm{u},j}$ is the j^{th} column of $\mathbf{C}_{\mathrm{u}}$, $\mathbf{r}_{\mathrm{u},i}^{\mathsf{T}}$ is the i^{th} row of $\mathbf{R}_{\mathrm{u}}$ and $\hat{\mathbf{a}}_{\mathrm{u},j}$ is the j^{th} column of $\mathbf{A}_{\mathrm{u}}$. The uplink SINR conditions can be expressed as

$$\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)\boldsymbol{\rho}_{\mathrm{c,u}} = \mathbf{J}_{\mathrm{u}}\boldsymbol{\gamma}\ . \tag{6.3}$$

Similarly, let $\mathbf{G}_{\mathrm{d}} \triangleq \operatorname{diag}([\|\mathbf{c}_{\mathrm{d},1}\|^2 \ \cdots \ \|\mathbf{c}_{\mathrm{d},L}\|^2])$ and $\mathbf{M}_{\mathrm{d}} \in \mathbb{R}^{L\times L}$, where $[\mathbf{M}_{\mathrm{d}}]_{ij} = (\mathbf{c}_{\mathrm{d},i}^{\mathsf{T}}\mathbf{H}_{\mathrm{d}}\mathbf{b}_{\mathrm{d},j} - \mathbf{a}_{\mathrm{d},i}^{\mathsf{T}}\mathbf{r}_{\mathrm{d},j})^2$. Here $\mathbf{a}_{\mathrm{d},i}^{\mathsf{T}}$ is the i^{th} row of $\mathbf{A}_{\mathrm{d}}$, $\mathbf{b}_{\mathrm{d},j}$ is the j^{th} column of $\mathbf{B}_{\mathrm{d}}$ and $\mathbf{r}_{\mathrm{d},j}$ is the j^{th} column of $\mathbf{R}_{\mathrm{d}}$. The downlink SINR conditions can be expressed as

$$\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)\boldsymbol{\rho}_{\mathrm{c,d}} = \mathbf{G}_{\mathrm{d}}\boldsymbol{\beta}\ . \tag{6.4}$$

Consider the transpose condition for all channel settings, such that $\mathbf{B}_{\mathrm{u}} = \mathbf{B}_{\mathrm{d}}^{\mathsf{T}}$, $\mathbf{A}_{\mathrm{u}} = \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}$, $\mathbf{H}_{\mathrm{u}} = \mathbf{H}_{\mathrm{d}}^{\mathsf{T}}$ and $\mathbf{C}_{\mathrm{u}} = \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}$. If we set $\mathbf{R}_{\mathrm{d}} = \mathbf{R}_{\mathrm{u}}^{\mathsf{T}}$, then $\mathbf{M}_{\mathrm{d}} = \mathbf{M}_{\mathrm{u}}^{\mathsf{T}}$. We now turn to show that there exists a positive power vector $\boldsymbol{\rho}_{\mathrm{c,d}}$ that achieves the desired downlink computation rates. By definition, $\mathbf{M}_{\mathrm{u}}$ is a non-negative matrix, thus, $\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)$ is a square Z-matrix. By assumption, the vector $\mathbf{J}_{\mathrm{u}}\boldsymbol{\rho}_{\mathrm{c,u}}$ is positive, which means that the uplink coding power vector $\boldsymbol{\rho}_{\mathrm{c,u}}$ satisfies condition ii) of Lemma 5. Therefore, $\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)$ is a non-singular M-matrix and all of its real eigenvalues are positive. Setting $\boldsymbol{\gamma} = \boldsymbol{\beta}$, we have

$$\begin{aligned}
\operatorname{eig}(\operatorname{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}) &= \operatorname{eig}(\operatorname{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{d}}^{\mathsf{T}}) \\
&= \operatorname{eig}(\mathbf{M}_{\mathrm{d}}\operatorname{diag}(\boldsymbol{\gamma})) \\
&= \operatorname{eig}(\mathbf{M}_{\mathrm{d}}\operatorname{diag}(\boldsymbol{\beta})) \\
&= \operatorname{eig}(\operatorname{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}})
\end{aligned}$$

This implies that every real eigenvalue of $\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)$ is also positive, which satisfies condition iii) of Lemma 5, and implies that $\Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)^{-1}$ exists and is non-negative. Since, by definition, $\mathbf{G}_{\mathrm{d}}\boldsymbol{\beta}$ is positive, we have that $\boldsymbol{\rho}_{\mathrm{c,d}} = \Big(\mathbf{I} - \operatorname{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)^{-1}\mathbf{G}_{\mathrm{d}}\boldsymbol{\beta}$ is non-negative. In summary, when $\boldsymbol{\gamma} = \boldsymbol{\beta}$, a valid power vector $\boldsymbol{\rho}_{\mathrm{c,d}}$ exists that satisfies (6.4). Now, we are ready to prove that the total uplink and downlink powers are equal when $\boldsymbol{\gamma} = \boldsymbol{\beta}$.

Define $\mathbf{G}_{\mathrm{u}} \triangleq \operatorname{diag}([\|\mathbf{c}_{\mathrm{u},1}\|^2 \ \cdots \ \|\mathbf{c}_{\mathrm{u},L}\|^2])$ and $\mathbf{J}_{\mathrm{d}}$ to be the $L\times L$ matrix with $(m,\ell)^{\mathrm{th}}$ entry $b_{\mathrm{d},m,\ell}^2$ where $b_{\mathrm{d},m,\ell}$ is the $(m,\ell)^{\mathrm{th}}$ of $\mathbf{B}_{\mathrm{d}}$. Let $\boldsymbol{\rho}_{\mathrm{t,u}} \triangleq \Big[\mathbb{E}\|\mathbf{x}_{\mathrm{u},1}\|^2 \cdots \mathbb{E}\|\mathbf{x}_{\mathrm{u},L}\|^2\Big]^{\mathsf{T}} = \mathbf{G}_{\mathrm{u}}\boldsymbol{\rho}_{\mathrm{c,u}}$ and $\boldsymbol{\rho}_{\mathrm{t,d}} \triangleq \Big[\mathbb{E}\|\mathbf{x}_{\mathrm{d},1}\|^2 \cdots \mathbb{E}\|\mathbf{x}_{\mathrm{d},L}\|^2\Big]^{\mathsf{T}} = \mathbf{J}_{\mathrm{d}}\boldsymbol{\rho}_{\mathrm{c,d}}$ denote the uplink and downlink transmit power vectors, respectively. Since $\Big(\mathbf{I} - \mathrm{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)$ and $\Big(\mathbf{I} - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)$ are both invertible, from (6.3) and (6.4) we have that

$$\boldsymbol{\rho}_{\mathrm{t,u}} = \mathbf{G}_{\mathrm{u}}\Big(\mathbf{I}_K - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)^{-1}\mathbf{J}_{\mathrm{u}}\boldsymbol{\gamma} \tag{6.5}$$

$$\boldsymbol{\rho}_{\mathrm{t,d}} = \mathbf{J}_{\mathrm{d}}\Big(\mathbf{I}_K - \mathrm{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)^{-1}\mathbf{G}_{\mathrm{d}}\boldsymbol{\beta}\ . \tag{6.6}$$

Finally, it can be shown that the total power in the downlink equals that in the uplink,

$$\begin{aligned}
P_{\mathrm{total}} =& \mathbf{1}^{\mathsf{T}}\mathbf{J}_{\mathrm{d}}\Big(\mathbf{I}_L - \mathrm{diag}(\boldsymbol{\beta})\mathbf{M}_{\mathrm{d}}\Big)^{-1}\mathbf{G}_{\mathrm{d}}\boldsymbol{\beta} \\
=& \mathbf{1}^{\mathsf{T}}\mathbf{J}_{\mathrm{d}}\Big(\mathbf{G}_{\mathrm{d}}^{-1}\,\mathrm{diag}(\boldsymbol{\beta})^{-1} - \mathbf{G}_{\mathrm{d}}^{-1}\mathbf{M}_{\mathrm{d}}\Big)^{-1}\mathbf{1} \\
=& \mathbf{1}^{\mathsf{T}}\Big(\,\mathrm{diag}(\boldsymbol{\beta})^{-1}\mathbf{G}_{\mathrm{d}}^{-1} - \mathbf{M}_{\mathrm{d}}^{\mathsf{T}}\mathbf{G}_{\mathrm{d}}^{-\mathsf{T}}\Big)^{-1}\mathbf{J}_{\mathrm{d}}^{\mathsf{T}}\mathbf{1} \\
=& \mathbf{1}^{\mathsf{T}}\Big(\,\mathrm{diag}(\boldsymbol{\beta})^{-1}\mathbf{G}_{\mathrm{d}}^{-1} - \mathbf{M}_{\mathrm{d}}^{\mathsf{T}}\mathbf{G}_{\mathrm{d}}^{-\mathsf{T}}\Big)^{-1}\mathbf{J}_{\mathrm{u}}\mathbf{1} \\
=& \mathbf{1}^{\mathsf{T}}\Big(\mathbf{J}_{\mathrm{u}}^{-1}\,\mathrm{diag}(\boldsymbol{\gamma})^{-1}\mathbf{G}_{\mathrm{d}}^{-1} - \mathbf{J}_{\mathrm{u}}^{-1}\mathbf{M}_{\mathrm{u}}\mathbf{G}_{\mathrm{d}}^{-\mathsf{T}}\Big)^{-1}\mathbf{1} \\
=& \mathbf{1}^{\mathsf{T}}\Big(\mathbf{J}_{\mathrm{u}}^{-1}\,\mathrm{diag}(\boldsymbol{\gamma})^{-1}\mathbf{G}_{\mathrm{u}}^{-1} - \mathbf{J}_{\mathrm{u}}^{-1}\mathbf{M}_{\mathrm{u}}\mathbf{G}_{\mathrm{u}}^{-1}\Big)^{-1}\mathbf{1} \\
=& \mathbf{1}^{\mathsf{T}}\mathbf{G}_{\mathrm{u}}\Big(\mathbf{I}_L - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{M}_{\mathrm{u}}\Big)^{-1}\mathbf{J}_{\mathrm{u}}\boldsymbol{\gamma} = P_{\mathrm{total}}\ .
\end{aligned}$$

So far we assume $\gamma_\ell \triangleq P_{\mathrm{u},\ell}/\sigma^2_{\mathrm{u,SIC},\ell}$ denote the SINR for the ℓ^{th} uplink user in (3.44), $\beta_\ell \triangleq P_{\mathrm{d},\ell}/\sigma^2_{\mathrm{d,DPC},\ell}$ denote the SINR for the ℓ^{th} downlink user in (4.62) and $\gamma_\ell = \beta_\ell, \forall\ell$. However, there is no guarantee that $\sigma^2_{\mathrm{u,SIC},1} \leq \cdots \leq \sigma^2_{\mathrm{u,SIC},L}$ in Definition 3 and $P_{\mathrm{d},1} \geq \cdots \geq P_{\mathrm{d},L}$ in Theorem 4 can be satisfied simultaneously. Thus, we need to re-assign the coding power to effective noise where the actual achievable computational rates are given by

$$R^{\mathrm{SIC}}_{\mathrm{u},m} = \frac{1}{2}\log^+\left(\frac{P_{\mathrm{u},m}}{\sigma^2_{\mathrm{u,SIC},\pi(m)}}\right)\ . \tag{6.7}$$

and

$$R^{\mathrm{DPC}}_{\mathrm{d},m} = \frac{1}{2}\log^+\left(\frac{P_{\mathrm{d},\theta(m)}}{\sigma^2_{\mathrm{d,DPC},m}}\right) \tag{6.8}$$

where π and θ are some valid permutation orders such that $\sigma^2_{\mathrm{SIC},\pi(1)} \leq \cdots \leq \sigma^2_{\mathrm{SIC},\pi(L)}$ and $P_{\mathrm{d},\theta(1)} \geq \cdots \geq P_{\mathrm{d},\theta(L)}$. If π and θ are identity permutations (which is not true in general), we have $R^{\mathrm{SIC}}_{\mathrm{u},m} = R^{\mathrm{DPC}}_{\mathrm{d},m}, \forall m$. If π and θ are not identity, we still achieve the same sum rate for both cases,

$$\sum_m R^{\mathrm{SIC}}_{\mathrm{u},m} = \frac{1}{2}\log\left(\frac{\prod_m P_{\mathrm{u},m}}{\prod_m \sigma^2_{\mathrm{u,SIC},\pi(m)}}\right) \tag{6.9}$$

$$= \frac{1}{2}\log\left(\frac{\prod_m P_{\mathrm{u},m}}{\prod_m \sigma^2_{\mathrm{u,SIC},m}}\right) \tag{6.10}$$

$$= \frac{1}{2}\log\left(\frac{\prod_m P_{\mathrm{d},m}}{\prod_m \sigma^2_{\mathrm{d,eff},m}}\right) \tag{6.11}$$

$$= \frac{1}{2}\log\left(\frac{\prod_m P_{\mathrm{d},\theta(m)}}{\prod_m \sigma^2_{\mathrm{d,eff},m}}\right) \tag{6.12}$$

$$= \sum_m R^{\mathrm{DPC}}_{\mathrm{d},m} \tag{6.13}$$

□

6.1.2 By-product #1: Iterative Optimization via Duality

Consider the following optimization problem where we focus on the sum-rate optimality.

$$\begin{aligned} \text{Uplink:}\quad & \arg\max \lim_{\mathbf{A}_\mathrm{u},\mathbf{C}_\mathrm{u},\mathbf{B}_\mathrm{u},\mathbf{R}_\mathrm{u},\boldsymbol{\rho}_{\mathrm{c,u}}} \quad \sum_{\ell=1}^{L}\frac{1}{2}\log^+(\gamma_\ell) \\ \text{s.t.}\quad & \mathrm{Tr}(\mathbf{C}_\mathrm{u}^\mathsf{T}\mathbf{C}_\mathrm{u}\mathbf{P}_\mathrm{u}) \le P_{\mathrm{total}} \end{aligned} \tag{6.14}$$

$$\begin{aligned} \text{Downlink:}\quad & \arg\max \lim_{\mathbf{A}_\mathrm{d},\mathbf{C}_\mathrm{d},\mathbf{B}_\mathrm{d},\mathbf{R}_\mathrm{d}\boldsymbol{\rho}_{\mathrm{c,d}}} \quad \sum_{\ell=1}^{L}\frac{1}{2}\log^+(\beta_\ell) \\ \text{s.t.}\quad & \mathrm{Tr}(\mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{B}_\mathrm{d}\mathbf{P}_\mathrm{d}) \le P_{\mathrm{total}} \end{aligned} \tag{6.15}$$

Recall that γ_ℓ and β_ℓ represent the ℓ^{th} effective SINR for the uplink and the downlink channel, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ represent the corresponding effective SINR vector, respectively. For the optimization problem in (6.14) and (6.15), even for a fixed integer matrix $\mathbf{A}_\mathrm{u}$ and $\mathbf{A}_\mathrm{d}$, optimizing the remaining parameters jointly is in general a non-convex problem. Below, we will take an iterative approach that converges to a local optimum. To this end, we first show how to find the optimal solution for each matrix in closed form, assuming that the remaining matrices are fixed.

In [Ordentlich et al., 2013], it was shown that it is optimal to set $\mathbf{R}_{\mathrm{u}}$ to

$$\mathbf{R}_{\mathrm{u,opt}} = \mathrm{diag}(f_{11}, \cdots, f_{LL})\mathbf{F}^{-1} \tag{6.16}$$

where $\mathbf{F}\mathbf{F}^{\mathsf{T}} = \mathbf{A}_{\mathrm{u}}(\mathbf{P}_{\mathrm{u}}^{-1} + \mathbf{H}_{\mathrm{u}}^{\mathsf{T}}\mathbf{C}_{\mathrm{u}}^{\mathsf{T}}\mathbf{C}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}})\mathbf{A}_{\mathrm{u}}^{\mathsf{T}}$ is the Cholesky factorization and $f_{\ell\ell}$ is the ℓ^{th} diagonal entry of $\mathbf{F}$.

Before we pass these choices to the dual downlink problem, we need to ensure that the effective SINRs correspond to an achievable sum rate. This will occur if and only if the effective noises are non-decreasing, $\sigma^2_{\mathrm{u,SIC},1} \leq \cdots \leq \sigma^2_{\mathrm{u,SIC},L}$. Assume that the m^{th} effective noise variance is the first to violate this condition, $\sigma^2_{\mathrm{u,SIC},m} < \sigma^2_{\mathrm{u,SIC},m-1}$. By replacing the successive cancellation vector $\mathbf{r}^{\mathsf{T}}_{\mathrm{u},m} = [r_{\mathrm{u},m,1} \ \cdots \ r_{\mathrm{u},m-1} \ 1 \ 0 \ \cdots 0]$ with the vector

$$\tilde{\mathbf{r}}^{\mathsf{T}}_{\mathrm{u},m} = [\alpha_m r_{\mathrm{u},m,1} \ \cdots \ \alpha_m r_{\mathrm{u},m-1} \ 1 \ 0 \ \cdots 0] \tag{6.17}$$

$$\alpha_m = \frac{-\mathbf{t}_m^{\mathsf{T}}\mathbf{f}_m + \sqrt{(\mathbf{t}_m^{\mathsf{T}}\mathbf{f}_m)^2 - \|\mathbf{t}_m\|^2\left(\|\mathbf{f}_m\|^2 - \sigma^2_{\mathrm{u,SIC},m-1}\right)}}{\|\mathbf{t}_m\|^2} \tag{6.18}$$

$$\mathbf{t}_m = \sum_{\ell=1}^{m-1} r_{\mathrm{u},m,\ell}\mathbf{f}_\ell^{\mathsf{T}} \tag{6.19}$$

where $\mathbf{f}_\ell^{\mathsf{T}}$ denotes the ℓ^{th} row of $\mathbf{F}$, the m^{th} effective noise variance will increase to $\sigma^2_{\mathrm{u,SIC},m} = \sigma^2_{\mathrm{u,SIC},m-1}$. Repeating this procedure in ascending order for all indices $m = 1, \ldots, L$, we will obtain a new choice of successive cancellation matrix $\tilde{\mathbf{R}}_{\mathrm{u}}$ that guarantees $\sigma^2_{\mathrm{u,SIC},1} \leq \cdots \leq \sigma^2_{\mathrm{u,SIC},L}$, and that the sum rate $\sum_\ell \frac{1}{2}\log^+\left(P_{\mathrm{u},\ell}/\sigma^2_{\mathrm{u,SIC},\ell}\right)$ is achievable.

The optimal equalization matrix $\mathbf{B}_{\mathrm{u}}$ is a quadratic problem with a closed-form

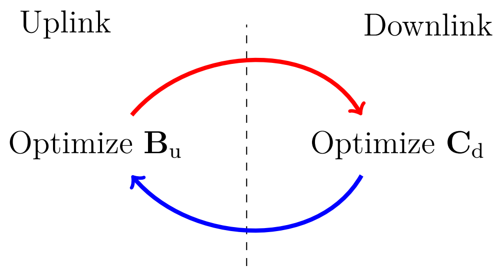

Figure 6·1: High-level overview of the iterative optimization algorithm for IF.

solution

$$\mathbf{B}_{\mathrm{u,opt}} = \mathbf{R}_{\mathrm{u}}\mathbf{A}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}}(\mathbf{I} + \mathbf{H}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}})^{-1}. \tag{6.20}$$

One efficient way to choose $\mathbf{A}_{\mathrm{u}}$ is given in [Zhan et al., 2014] by applying the LLL algorithm [Lenstra et al., 1982] to the matrix $\mathbf{V}\mathbf{D}^{\frac{1}{2}}$, where $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{T}}$ is the singular value decomposition of $\mathbf{H}_{\mathrm{u}}\mathbf{C}_{\mathrm{u}}$, and we define $\mathbf{D} = \mathbf{I} - \boldsymbol{\Sigma}^{\mathsf{T}}(\mathbf{P}_{\mathrm{u}}^{-1} + \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}$.

For problem (6.15) in the downlink channel, if fixed $\mathbf{A}_{\mathrm{d}}$, $\mathbf{R}_{\mathrm{d}}$, $\mathbf{B}_{\mathrm{d}}$, $\mathbf{H}_{\mathrm{d}}$ and $\boldsymbol{\rho}_{\mathrm{c,d}}$, optimal $\mathbf{c}_{\mathrm{d},m}$ is in a closed-form solution as:

$$\mathbf{c}_{\mathrm{d},m}^{\mathsf{T}} = \mathbf{a}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{R}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}^{\mathsf{T}}(\mathbf{I} + \mathbf{H}_{\mathrm{d},m}^{\mathsf{T}}\mathbf{B}_{\mathrm{d}}\mathbf{P}_{\mathrm{d}}\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{H}_{\mathrm{d},m}^{\mathsf{T}})^{-1}\ . \tag{6.21}$$

We introduce two iterative algorithms that include the SIC and DPC techniques for IF to improve the transmission rates in the uplink and the downlink channel, respectively. Both algorithms run in polynomial-time.

The duality relationship developed in this work suggests a simple iterative algorithm for optimizing the uplink sum rate. For instance, in the uplink, we can first solve for $\mathbf{B}_{\mathrm{u,opt}}^{(1)}$ in closed form for $\mathbf{c}_{\mathrm{u},\ell}^{(1)} = \mathbf{1}_{M_\ell}$ for all ℓ. Then, we create a virtual downlink channel, solve for $\mathbf{C}_{\mathrm{d,opt}}^{(1)}$ and bring it back to the uplink channel via the duality relationship for the next iteration $\mathbf{C}_{\mathrm{u}}^{\mathsf{T}(2)} = \mathbf{C}_{\mathrm{d,opt}}^{(1)}$.

Here, we set $\mathbf{R}_{\mathrm{u}}$ to be identity at the beginning and double check it after the iteration complete to guarantee $\sigma^2_{\mathrm{u,SIC},1} \le \sigma^2_{\mathrm{u,SIC},2} \le \cdots \le \sigma^2_{\mathrm{u,SIC},L}$ is satisfied.

Algorithm 1 Iterative Uplink Optimization via Duality

Given $\mathbf{H}_\mathrm{u}$ and P_total
Initialization Step:
Set initial parameters $\mathbf{A}_\mathrm{u}$, $\mathbf{R}_\mathrm{u}$, $\mathbf{B}_\mathrm{u}$, $\boldsymbol{\rho}_\mathrm{c,u}$ and $\mathbf{C}_\mathrm{u}$.
Calculate $\boldsymbol{\gamma}$.
while $\boldsymbol{\gamma}$ not converged **do**
  Create virtual dual downlink channel with $\mathbf{A}_\mathrm{d} = \mathbf{A}_\mathrm{u}^\mathsf{T}$, $\mathbf{B}_\mathrm{d} = \mathbf{B}_\mathrm{u}^\mathsf{T}$, $\mathbf{C}_\mathrm{d} = \mathbf{C}_\mathrm{u}^\mathsf{T}$, $\mathbf{R}_\mathrm{d} = \mathbf{R}_\mathrm{u}^\mathsf{T}$ and $\beta_\ell = \gamma_\ell$.
  Calculate $\boldsymbol{\rho}_\mathrm{c,d}$ using (6.4).
  Optimize $\mathbf{C}_\mathrm{d}$ using (6.21).
  Update $\boldsymbol{\beta}$.
  Update $\mathbf{C}_\mathrm{u} = \mathbf{C}_\mathrm{d}^\mathsf{T}$ and $\beta_\ell = \gamma_\ell$.
  Calculate $\boldsymbol{\rho}_\mathrm{c,u}$ using (6.3).
  Update $\mathbf{R}_\mathrm{u}$ using (6.16)
  for $m = 2$ **to** L **do**
    if $\sigma^2_{\mathrm{u,SIC},m} < \sigma^2_{\mathrm{u,SIC},m-1}$ **then**
      Set $\mathbf{r}_{\mathrm{u},m} = \tilde{\mathbf{r}}_{\mathrm{u},m}$ using (6.17).
    end if
  end for
  Optimize $\mathbf{B}_\mathrm{u}$ using (6.20).
  Update $\boldsymbol{\gamma}$.
end while
Output $\mathbf{A}_\mathrm{u}$, $\mathbf{B}_\mathrm{u}$, $\mathbf{C}_\mathrm{u}$, $\mathbf{R}_\mathrm{u}$, $\boldsymbol{\rho}_\mathrm{c,u}$, and $\boldsymbol{\gamma}$.

The iterative algorithm for optimizing the downlink sum rate is a slight variation of Algorithm 1. The process is summarized in Algorithm 2. It is quite similar to that for the uplink, so we omit a detailed description due to space constraints.

Algorithm 2 Iterative Downlink Optimization via Duality

Given $\mathbf{H}_{\mathrm{d}}$ and P_{total}
Initialization Step:
Set initial parameters $\mathbf{A}_{\mathrm{d}}$, $\mathbf{R}_{\mathrm{d}}$, $\mathbf{B}_{\mathrm{d}}$, $\boldsymbol{\rho}_{\mathrm{c,d}}$ and $\mathbf{C}_{\mathrm{d}}$.
Calculate $\boldsymbol{\beta}$.
while $\boldsymbol{\beta}$ not converged **do**
  Create virtual dual uplink channel with $\mathbf{A}_{\mathrm{u}} = \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}$, $\mathbf{B}_{\mathrm{u}} = \mathbf{B}_{\mathrm{d}}^{\mathsf{T}}$, $\mathbf{C}_{\mathrm{u}} = \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}$, $\mathbf{R}_{\mathrm{u}} = \mathbf{R}_{\mathrm{d}}^{\mathsf{T}}$ and $\beta_\ell = \gamma_\ell$.
  Calculate $\boldsymbol{\rho}_{\mathrm{c,u}}$ using (6.3).
  Update $\mathbf{R}_{\mathrm{u}}$ using (6.16).
  for $m = 2$ **to** L **do**
    if $\sigma^2_{\mathrm{u,SIC},m} < \sigma^2_{\mathrm{u,SIC},m-1}$ **then**
      Set $\mathbf{r}_{\mathrm{u},m} = \tilde{\mathbf{r}}_{\mathrm{u},m}$ using (6.17).
    end if
  end for
  Optimize $\mathbf{B}_{\mathrm{u}}$ using (6.20).
  Update $\boldsymbol{\gamma}$.
  Update $\mathbf{B}_{\mathrm{d}} = \mathbf{B}_{\mathrm{u}}^{\mathsf{T}}$, $\mathbf{R}_{\mathrm{d}} = \mathbf{R}_{\mathrm{u}}^{\mathsf{T}}$ and $\beta_\ell = \gamma_\ell$.
  Calculate $\boldsymbol{\rho}_{\mathrm{c,d}}$ using (6.4).
  Optimize $\mathbf{C}_{\mathrm{d}}$ using (6.21).
  Update $\boldsymbol{\beta}$.
end while
Output $\mathbf{A}_{\mathrm{d}}$, $\mathbf{B}_{\mathrm{d}}$, $\mathbf{C}_{\mathrm{d}}$, $\mathbf{R}_{\mathrm{d}}$, $\boldsymbol{\rho}_{\mathrm{c,d}}$, and $\boldsymbol{\beta}$.

We now briefly investigate the performance of our iterative algorithm for the uplink setting. To generate the plot, we have set $L = 5$, and drawn a sample channel matrix $\mathbf{H}_{\mathrm{u}}$ elementwise i.i.d. $\mathcal{N}(0, 1)$. For simplicity, we assume single-antenna user case. The integer matrix $\mathbf{A}_{\mathrm{u}}$ is initially chosen using the LLL algorithm and then remains fixed. In Figure 6·2, we have plotted the effective SINRs obtained after a given number of iterations by Algorithm 1. Here we set $\mathbf{R}_{\mathrm{u}}$ and $\mathbf{R}_{\mathrm{d}}$ to be identities. Figure 6·2 shows all SINRs increase monotonically after each iteration.

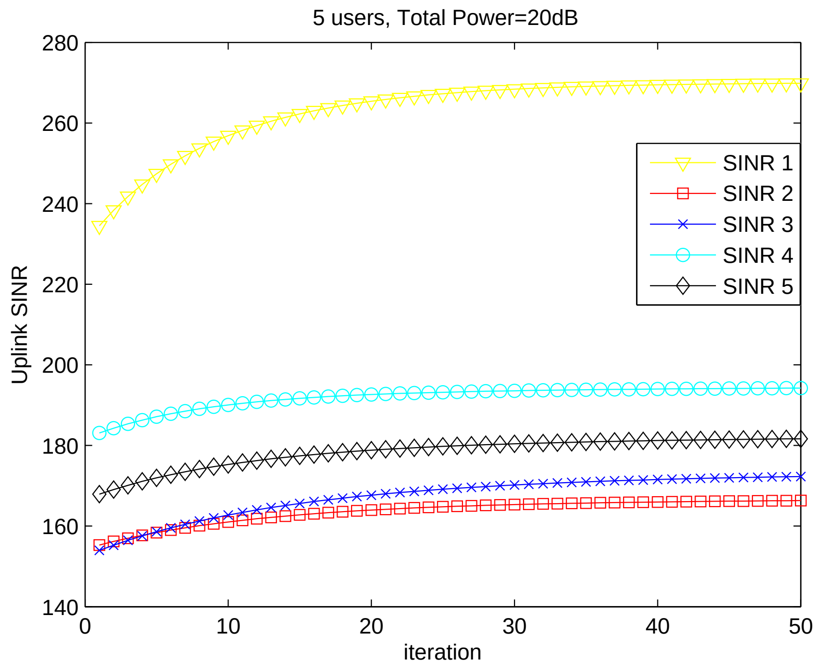

Figure 6·2: Achievable SINRs for 5-user uplink channel after a given number of iterations using Algorithm 1 with fixed total power $P_{\text{total}} =$ 20dB.

In Figure 6·3, we plot the symmetric rate (the minimum rate across all users) achieved by Algorithm 1 and compare it with three other schemes. In Figure 6·3, ZF means we set $\mathbf{A}_{\text{u}}$ and $\mathbf{A}_{\text{d}}$ to be identity matrices and "no SIC" means we fix $\mathbf{R}_{\text{u}}$ and $\mathbf{R}_{\text{d}}$ to be identity matrices.

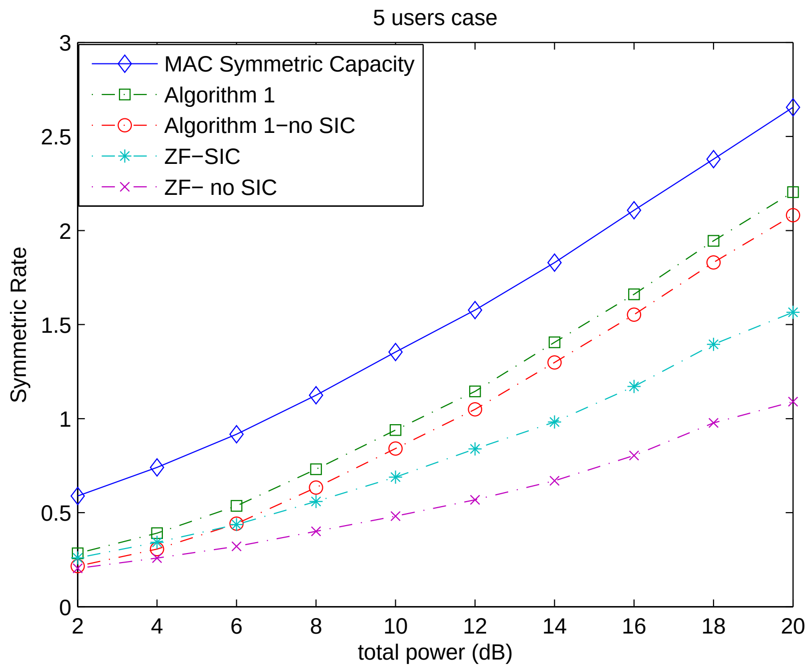

Figure 6·3: Performance of iterative optimization algorithm for IF using uplink-downlink duality. Here we comparison the symmetric rates for 5 users in the uplink MIMO MAC.

6.1.3 By-product #2: Peak-to-average Power Ratio for Downlink IF

In [Nazer and Gastpar, 2011] and [Zhan et al., 2014], integer-forcing shows its advantage in the sense of uplink computation rates, which can be extended into the dual downlink channel using the uplink-downlink duality relationship. The base station in the downlink channel has multiple antennas and the transmission is under a total power constraint. The power can be highly unbalanced across base station antennas, which in turn leads to large peak single antenna power and results in a high implementation cost for the power amplifier.

In Figure 6·4, we have plotted the peak-to-average power ratio (PAPR) for all of the strategies described above in the context of the downlink. Specifically, we plot the highest power across the transmit antennas divided by the average power. For ZF, we fix $\mathbf{A}_{\mathrm{d}}$ to be identity. For no DPC case, we fix $\mathbf{R}_{\mathrm{d}}$ to be identity. We consider a downlink channel with $N = 4$ basestation antennas and $L = 4$ single-antenna users. Each channel matrix $\mathbf{H}_{\mathrm{d}}$ is drawn elementwise i.i.d. $\mathcal{N}(0, 1)$ and our result is averaged over 2000 independent channel realizations. Note that integer-forcing performs similarly to zero-forcing with DPC and that integer-forcing with DPC has a lower PAPR than zero-forcing with DPC.

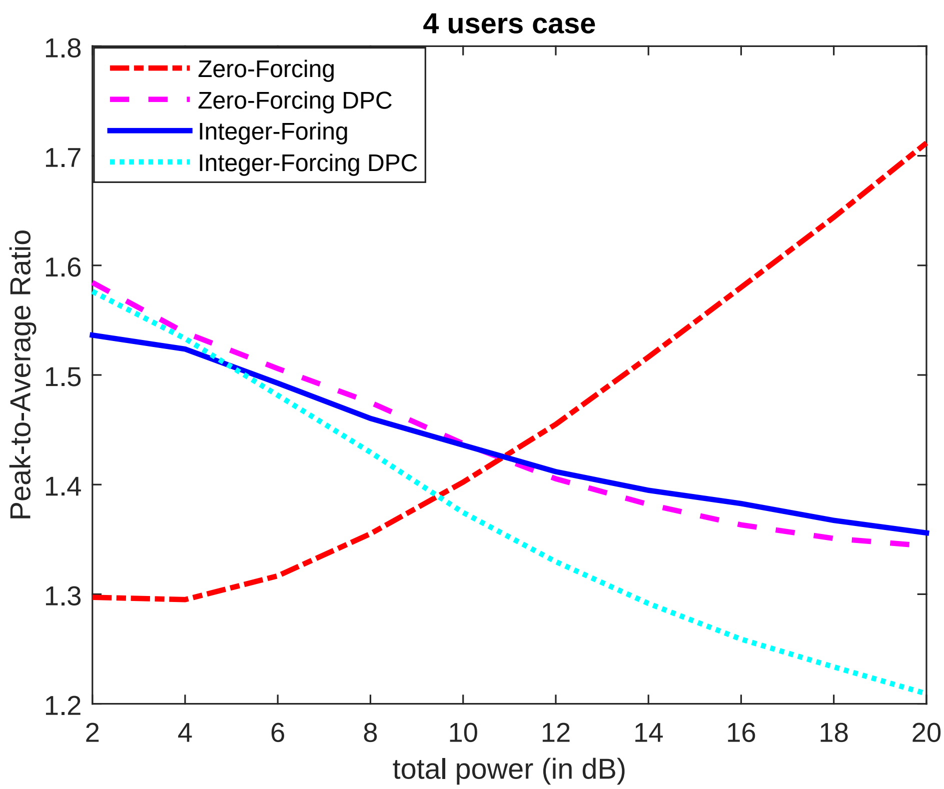

Figure 6·4: Peak-to-average power ratio for integer-forcing and zero-forcing architectures for a downlink channel with $N = 4$ basestation antennas and $L = 4$ single-antenna users.

6.1.4 By-product #3: Broadcast Sum Capacity to within a Constant Gap for IF without DPC

By using the integer-forcing uplink-downlink duality and existing result from [Ordentlich et al., 2012], we show that IF (without DPC) can achieve rate tuples within a constant gap of the sum capacity in the downlink Gaussian BC channel.

Previous works in [Ordentlich et al., 2012] show that IF can can approach the sum capacity of the L-user multiple-access channel within a constant gap of $\frac{L}{2}\log_2 L$ bits. In [Ordentlich et al., 2012], the coding powers are assumed to be symmetric, and transmitters are assumed to be single-antenna without beamforming ($\mathbf{C}_\mathrm{u}$ is identity). We will first review the results in [Nazer et al., 2016] which generalize [Ordentlich et al., 2012][Theorem 3] to asymmetric power case. For simplicity, we will assume single-antenna user case where multi-antenna user case can be transfered into a single-antenna case using unitary beamforming vectors in the uplink and unitary projection vectors in the downlink.

Lemma 6 ([Nazer et al., 2016]). *For a given uplink channel matrix $\mathbf{H}_\mathrm{u} \in \mathbb{R}^{N\times L}$ and coding power matrix $\mathbf{P}_\mathrm{u}$, if $\mathbf{C}_\mathrm{u}$ and $\mathbf{R}_\mathrm{u}$ are set to be identity (no beamforming and no SIC), the optimal sum computation rate (optimized over $\mathbf{A}_\mathrm{u}$ and $\mathbf{B}_\mathrm{u}$) is lower bounded by*

$$\sum_{\ell=1}^{L} R_{\mathrm{u},\ell} \geq \frac{1}{2}\log_2 \det\left(\mathbf{I} + \mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\mathsf{T}\mathbf{H}_\mathrm{u}\right) - \frac{L}{2}\log_2 L \tag{6.22}$$

where the form of computation rate $R_{\mathrm{u},\ell}$ is given in (3.19).

The proof of Lemma 6 is given in [Nazer et al., 2016] which is also shown in Appendix B.

Theorem 10. *For a given downlink channel matrix $\mathbf{H}_\mathrm{d} \in \mathbb{R}^{L\times N}$ and (diagonal) equalization matrix $\mathbf{C}_\mathrm{d} = \mathbf{I}$, there exists a set of integer matrix $\mathbf{A}_\mathrm{d} \in \mathbb{Z}^{L\times L}$, beamforming matrix $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{N\times L}$ and (diagonal) power matrix $\mathbf{P}_\mathrm{d}$ such that the sum computation rate $\sum_{\ell=1}^{L} R_{\mathrm{d},\ell}$ approaches the sum capacity up to a constant gap of no more than $\frac{L}{2}\log_2 L$. Here $R_{\mathrm{d},\ell}$ is given in Theorem 4 (no DPC).*

Proof of Theorem 10: Based on [Vishwanath et al., 2003], [Viswanath and Tse, 2003] and [Yu and Cioffi, 2004], the sum capacity of a Gaussian BC channel and its dual Gaussian MAC channel are the same. For a given dual Gaussian MAC channel $\mathbf{H}_\mathrm{u} = \mathbf{H}_\mathrm{d}^\mathsf{T}$, there exists a power matrix $\mathbf{P}_\mathrm{u}^*$ that achieves the sum capacity (total power constraint).

$$\begin{aligned} \mathbf{P}_{\mathrm{u}}^{*} = \arg\max_{\mathbf{P}_{\mathrm{u}}^{*}} \quad & \frac{1}{2}\log_2 \det\left(\mathbf{I} + \mathbf{P}_{\mathrm{u}}^{*}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}\right) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{P}_{\mathrm{u}}^{*}) \le P_{\mathrm{total}} \end{aligned} \tag{6.23}$$

For that specific $\mathbf{P}_{\mathrm{u}}^{*}$, Lemma 6 shows that up to a constant gap of no more than $\frac{L}{2}\log_2 L$ can be approached to the sum capacity by fixing $\mathbf{C}_{\mathrm{u}}$, $\mathbf{R}_{\mathrm{u}}$ to be identity (no beamforming and SIC) and select $\mathbf{A}_{\mathrm{u}}$, $\mathbf{B}_{\mathrm{u}}$ optimally . Using the uplink-downlink duality result from Theorem 8, there exist a power matrix $\mathbf{P}_{\mathrm{d}}^{*}$ such that same sum rate can also be achieved by setting $\mathbf{C}_{\mathrm{d}}$, $\mathbf{R}_{\mathrm{d}}$ to be identity (no projection and DPC) and $\mathbf{A}_{\mathrm{d}} = \mathbf{A}_{\mathrm{u}}^{\mathsf{T}}$, $\mathbf{B}_{\mathrm{d}} = \mathbf{B}_{\mathrm{u}}^{\mathsf{T}}$. According to the uplink-downlink duality results, $\mathbf{P}_{\mathrm{d}}^{*}$ will not violate the total power constraint. $\square$

6.2 Source-Channel Duality for IF

In this section we will establish the source-channel duality for IF. Some notations will be changed from previous sections due to new variables and definitions. Specifically in this section, subscripts(or superscripts) “c” and “s” will be used to denote variables associated with the channel coding and source coding, respectively.

6.2.1 Review of Achievable Rate Expressions

We first briefly review achievable rates expressions for both IF channel coding and IF source coding in Chapter 3 and Chapter 5. For IF MAC channel coding, we set $\mathbf{C}_{\mathrm{u}} = \mathbf{I}$ representing identity beamforming. Without successive cancellation, the integer-forcing channel coding scheme consists of three steps: the receiver applies an equalization matrix, decodes integer-linear combinations of the transmitted lattice codewords, and then solves these linear combinations for the desired codewords. Select a full-rank integer matrix $\mathbf{A}_{\mathrm{c}} \in \mathbb{Z}^{K\times K}$ for which there exists a unit lower triangular matrix $\mathbf{L}$ such that $\mathbf{L}\mathbf{A}_{\mathrm{c}}$ is upper triangular.[1] Let $\mathbf{P} \triangleq \mathrm{diag}(P_1, \ldots, P_K)$ be the

[1] We can always find such an $\mathbf{L}$ for a full rank $\mathbf{A}_{\mathrm{c}}$ so long as we are permitted a column permutation on $\mathbf{A}_{\mathrm{c}}$, which is equivalent to reindexing the users.

diagonal matrix of power constraints as in Chapter 3 and let $\mathbf{a}_{\mathrm{c},k}^{\mathsf{T}}$ be the k^{th} row of $\mathbf{A}_{\mathrm{c}}$. We set equalization matrix $\mathbf{B}_{\mathrm{u}} = \mathbf{A}_{\mathrm{c}}\mathbf{P}^{\mathsf{T}}\mathbf{H}^{\mathsf{T}}(\mathbf{I}+\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{T}})^{-1}$ as in Chapter 3 where $\mathbf{H}$ is the uplink channel matrix. The following rates are achievable

$$R_k^{\mathrm{c}} = \frac{1}{2}\log\left(\frac{P_k}{\mathbf{a}_{\mathrm{c},k}^{\mathsf{T}}\left(\mathbf{H}^{\mathsf{T}}\mathbf{H}+\mathbf{P}^{-1}\right)^{-1}\mathbf{a}_{\mathrm{c},k}}\right) \tag{6.24}$$

for $k = 1,\dots,K$ as shown in [Ordentlich et al., 2013,Nazer et al., 2016] and Chapter 3.

For successive cancellation, we also select a unit, lower-triangular matrix $\mathbf{R}_{\mathrm{c}}$ whose k^{th} row we denote by $\mathbf{r}_{\mathrm{c},k}^{\mathsf{T}}$. This matrix must place the effective noises terms $\sigma_{\mathrm{eff},k}^2 = \mathbf{r}_{\mathrm{c},k}^{\mathsf{T}}\mathbf{A}_{\mathrm{c}}\left(\mathbf{H}^{\mathsf{T}}\mathbf{H}+\mathbf{P}^{-1}\right)^{-1}\mathbf{A}_{\mathrm{c}}^{\mathsf{T}}\mathbf{r}_{\mathrm{c},k}$ in increasing order, i.e., $\sigma_{\mathrm{eff},1}^2 \leq \cdots \leq \sigma_{\mathrm{eff},K}^2$. The successive integer-forcing strategy uses previously recovered linear combinations to reduce the effective noise variance encountered for decoding subsequent ones, according to the coefficients from $\mathbf{R}_{\mathrm{c}}$. Overall, as argued in [Ordentlich et al., 2013,Nazer et al., 2016] and Chapter 3, the following rates are achievable,

$$R_k^{\mathrm{c}} = \frac{1}{2}\log\left(\frac{P_k}{\mathbf{r}_{\mathrm{c},k}^{\mathsf{T}}\mathbf{A}_{\mathrm{c}}\left(\mathbf{H}^{\mathsf{T}}\mathbf{H}+\mathbf{P}^{-1}\right)^{-1}\mathbf{A}_{\mathrm{c}}^{\mathsf{T}}\mathbf{r}_{\mathrm{c},k}}\right) \tag{6.25}$$

for $k = 1,\dots,K$.

For source coding rate expression we use the formula given by Theorem 7 in Chapter 5.

6.2.2 Constant Gap Results without Successive Cancellation

We begin by considering integer-forcing without successive cancellation. For channel coding, the rates are given by (6.24) or, equivalently, by

$$R_k^{\mathrm{c}} = \frac{1}{2}\log(P_k) - \frac{1}{2}\log(\|\mathbf{G}\mathbf{a}_{\mathrm{c},k}\|^2) \tag{6.26}$$

where $\mathbf{G} = (\mathbf{H}^{\mathsf{T}}\mathbf{H}+\mathbf{P}^{-1})^{-\frac{1}{2}}$.

For source coding, the rates are given by Theorem 7 with $\mathbf{R} = \mathbf{I}$. Let $\mathbf{A}_{\mathrm{s}}$ be the integer matrix in IF source coding and let $\mathbf{a}_{\mathrm{s},k}^{\mathsf{T}}$ be the k^{th} row of $\mathbf{A}_{\mathrm{s}}$, we have the rate expression

$$R_k^{\mathrm{s}} = \frac{1}{2}\log\left(\frac{\mathbf{a}_{\mathrm{s},k}^{\mathsf{T}}(\mathbf{K}_{xx}+\mathbf{D})\mathbf{a}_{\mathrm{s},k}}{D_k}\right). \tag{6.27}$$

Setting $\mathbf{K}_{xx} = \mathbf{H}^{\mathsf{T}}\mathbf{H}$ and $P_k = D_k^{-1}$, we get that

$$R_k^{s} = \frac{1}{2}\log\left(||\mathbf{G}^{-\mathsf{T}}\mathbf{a}_{\mathrm{s},k}||^2\right) + \frac{1}{2}\log(P_k) \tag{6.28}$$

We now review some useful definitions and properties of successive minima and dual lattices.

Definition 5 (Successive Minima)**.** *Let* $\Lambda(\mathbf{G}) = \mathbf{G}\mathbb{Z}^K$ *be a* K*-dimensional lattice generated by* $\mathbf{G} \in \mathbb{R}^{K\times K}$*. For* $k = 1,\ldots,K$*, we define the* k^{th} *successive minima as*

$$\lambda_k \triangleq \inf\left\{r : \dim\left(\operatorname{span}\left(\Lambda(\mathbf{G}) \cap \mathcal{B}(\mathbf{0},r)\right)\right) \geq k\right\}$$

where $\mathcal{B}(\mathbf{0},r)$ *is a closed ball with radius* r *centered at* $\mathbf{0}$*.*

Definition 6 (Dual Lattice)**.** *The dual lattice of* $\Lambda(\mathbf{G})$ *is given by* $\Lambda(\mathbf{G}^{-\mathsf{T}})$*. We refer to the* k^{th} *successive minima of* $\Lambda(\mathbf{G}^{-\mathsf{T}})$ *as the* k^{th} *dual successive minima of* $\Lambda(\mathbf{G})$*, denoted as* λ_k^* *for* $k = 1,\ldots,K$*.*

The following result of Banaszczyk [Banaszczyk, 1993] connects the successive minima with their duals,

$$1 \leq \lambda_k \lambda_{K-k+1}^* \leq K, \quad k = 1,\ldots,K\ . \tag{6.29}$$

For the choice $\mathbf{G} = (\mathbf{H}^{\mathsf{T}}\mathbf{H} + \mathbf{P}^{-1})^{-\frac{1}{2}}$, the successive minima correspond to the optimal choices of integer vectors, leading to integer-forcing channel coding rates (6.24)

$$R_k^{\mathrm{c}} = \frac{1}{2}\log\left(\frac{1}{\lambda_k^2}\right) + \frac{1}{2}\log(P_k)\ . \tag{6.30}$$

For integer-forcing source coding, we use the successive minima in reverse order in order to satisfy (5.12), leading to achievable rates

$$R_k^{\mathrm{s}} = \frac{1}{2}\log((\lambda^*_{K-k+1})^2) + \frac{1}{2}\log(P_k) \tag{6.31}$$

Combining (6.29) with (6.31) and (6.30), we obtain the following bound

Theorem 11. *The optimal integer-forcing source coding and channel coding rates (without successive cancellation) lie within a constant gap of one another:*

$$R_k^c \leq R_k^s \leq R_k^c + \log K \quad k = 1, \ldots, K. \tag{6.32}$$

6.2.3 Successive IF Source-Channel Duality

Consider a successive integer-forcing channel coding scheme with integer matrix $\mathbf{A}_{\mathrm{c}}$ and lower triangular successive cancellation matrix $\mathbf{R}_{\mathrm{c}}$ satisfying the constraints from Chapter 3. Let $\mathbf{F}_{\mathrm{c}}\mathbf{F}_{\mathrm{c}}^{\mathsf{T}}$ be the Cholesky decomposition of $\mathbf{A}_{\mathrm{c}}(\mathbf{H}^{\mathsf{T}}\mathbf{H} + \mathbf{P}^{-1})^{-1}\mathbf{A}_{\mathrm{c}}^{\mathsf{T}}$ and $f_{\mathrm{c},k,k}$ be the k^{th} diagonal component of $\mathbf{F}_{\mathrm{c}}$. Setting $\mathbf{R}_{\mathrm{c}} = \operatorname{diag}(f_{\mathrm{c},1,1}, \ldots, f_{\mathrm{c},K,K})\mathbf{F}_{\mathrm{c}}^{-1}$, we can rewrite (6.25) as

$$R_k^{\mathrm{c}} = \frac{1}{2}\log(P_k) - \frac{1}{2}\log(f^2_{\mathrm{c},k,k}) \quad k = 1, \ldots, K. \tag{6.33}$$

We have that $f^2_{\mathrm{c},1,1} \leq \cdots \leq f^2_{\mathrm{c},K,K}$ (since the effective noises are assumed to be increasing).

Consider a successive integer-forcing source coding scheme with integer matrix $\mathbf{A}_{\mathrm{s}} = \mathbf{A}_{\mathrm{c}}^{-\mathsf{T}}$, upper triangular successive cancellation matrix

$$\mathbf{R}_{\mathrm{s}} = \mathbf{R}_{\mathrm{c}}^{-\mathsf{T}} = \operatorname{diag}(f^{-1}_{\mathrm{c},1,1}, \ldots, f^{-1}_{\mathrm{c},K,K})\mathbf{F}_{\mathrm{c}}^{\mathsf{T}} \tag{6.34}$$

, covariance matrix $\mathbf{K}_{xx} = \mathbf{H}^\mathsf{T}\mathbf{H}$, and distortions $D_k = P_k^{-1}$. It follows that

$$\mathbf{F}_\mathrm{c}^{-\mathsf{T}}\mathbf{F}_\mathrm{c}^{-1} = \Big(\mathbf{A}_\mathrm{c}\Big(\mathbf{H}^\mathsf{T}\mathbf{H} + \mathbf{P}^{-1}\Big)^{-1}\mathbf{A}_\mathrm{c}^\mathsf{T}\Big)^{-1} \tag{6.35}$$

$$= \mathbf{A}_\mathrm{c}^{-\mathsf{T}}(\mathbf{H}^\mathsf{T}\mathbf{H} + \mathbf{P}^{-1})\mathbf{A}_\mathrm{c}^{-1} \tag{6.36}$$

$$= \mathbf{A}_\mathrm{s}(\mathbf{K}_{xx} + \mathbf{D})\mathbf{A}_\mathrm{s}^\mathsf{T} \ . \tag{6.37}$$

Therefore, $\mathbf{r}_{\mathrm{s},m}^\mathsf{T}\mathbf{A}_\mathrm{s}(\mathbf{K}_{xx} + \mathbf{D})\mathbf{A}_\mathrm{s}^\mathsf{T}\mathbf{r}_{\mathrm{s},m} = f_{\mathrm{c},m,m}^{-2}$. Since $f_{\mathrm{c},1,1}^{-2} \geq \cdots \geq f_{\mathrm{c},K,K}^{-2}$, we have satisfied (5.12). It can also be shown that the desired submatrices are full rank. Overall, the achievable rate from Theorem 7 is

$$R_k^\mathrm{s} = \frac{1}{2}\log(P_k) - \frac{1}{2}\log(f_{\mathrm{c},k,k}^2) \quad k = 1, \ldots, K. \tag{6.38}$$

Combining (6.33) and (6.38), we have source-channel duality for IF established.

Theorem 12. *For the choices of integer and successive cancellation matrices above, successive integer-forcing source coding* (5.26) *and successive integer-forcing channel coding* (6.25) *achieve the same rates,*

$$R_k^\mathrm{s} = R_k^\mathrm{c} \tag{6.39}$$

for $k = 1, \ldots, K$.

Furthermore, it has been shown [Ordentlich et al., 2013] that successive integer-forcing achieves the multiple-access sum capacity when $\mathbf{R}_\mathrm{c} = \mathrm{diag}(f_{\mathrm{c},1,1}, \ldots, f_{\mathrm{c},K,K})\mathbf{F}_\mathrm{c}^{-1}$,

$$\sum_{k=1}^{K} R_k^\mathrm{c} = \frac{1}{2}\log\det(\mathbf{I} + \mathbf{P}\mathbf{H}^\mathsf{T}\mathbf{H}). \tag{6.40}$$

We can also conclude that the sum rate for IF channel coding and source coding both approach to the multiple-access channel coding sum capacity when the duality is established.

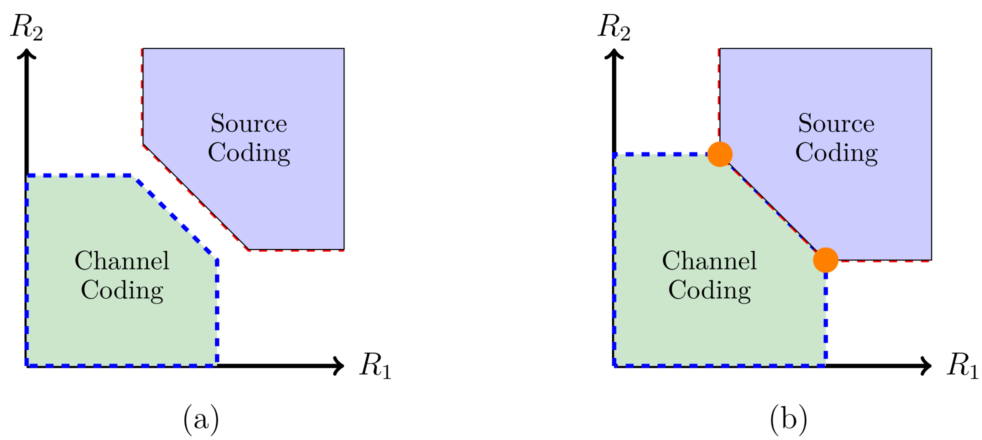

Figure 6·5: (a) Constant Gap without SIC; and (b) Source-Channel Duality with SIC.

Chapter 7

Integer-Forcing Interference Alignment (IFIA)

7.1 Overview

We first provide a high-level overview for integer-forcing interference alignment (IFIA). Consider a Gaussian MIMO interference channel with K transmitter-receiver pairs. Each pair will communicate using the same bandwidth and experience interference from other pairs. Let $x_1, x_2, \ldots, x_K$ denote the transmitted signals. We can align the signals such that at receiver k, the $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K$ occupy only half the signaling dimensions. We can then project onto the null space of the interfering signals which is the original interference alignment scheme in [Cadambe and Jafar, 2008, Jafar, 2011]. This resembles, at a high level, the zero-forcing approach for decoding multiple data streams over a point-to-point MIMO channel. For integer-forcing interference alignment, we choose to align signals in two (or multiple) signal-space directions. For example, on one direction we decode $x_k + f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K)$ and on the other direction we decode $x_k - f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K)$. Here $f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K)$ is a linear function of $\{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K\}$ with integer coefficients. In this case, interference is decoded as an aligned function instead of being nulled out. We can solve these two functions for x_k and repeat this process for the rest of the receivers and their desired signals. We can also choose to align interference in multiple directions and decode more than two combinations.

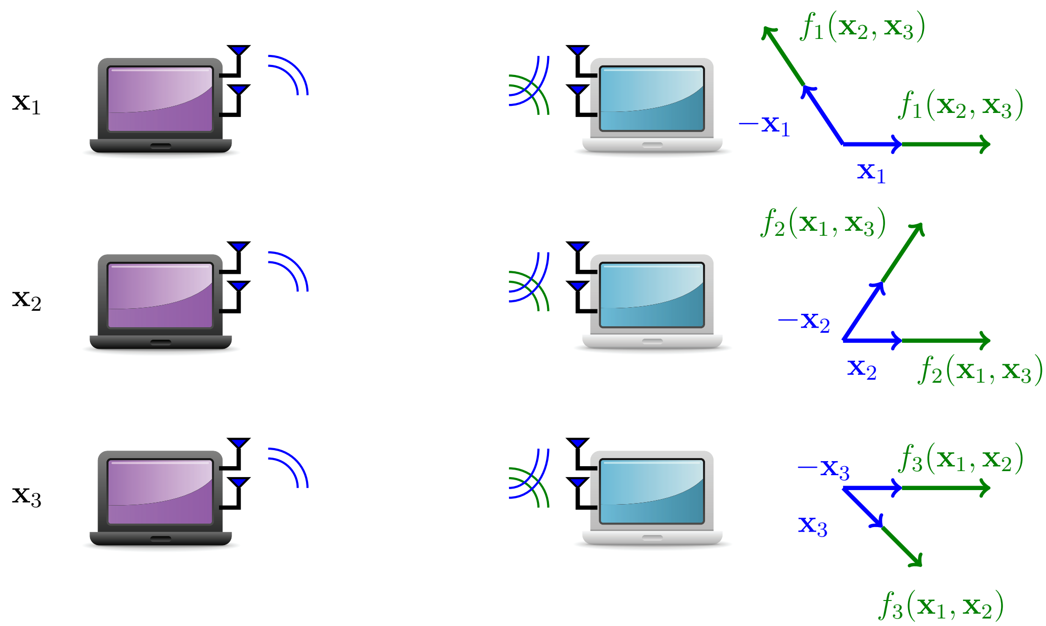

Figure 7·1: IFIA for the 3-user symmetric interference channel.

The scheme of integer-forcing interference alignment involves designing beamforming matrices and equalization matrices, as well as choosing the proper decoding combinations. In this chapter, we would like to answer the following questions: which combinations should we choose, how should we design beamfoming matrices and equalization matrices, and what is the performance of integer-forcing interference alignment at low/moderate SNR.

7.2 Problem Statement

We will focus on the K-user, real-valued Gaussian interference channel where each transmitter has a single data stream. Our framework can be naturally generalized to include multiple data streams per transmitters as well as any complex-valued channel by using its real-valued decomposition. Superscripts will refer to transmitter and receiver indices.

Transmitters: There are K transmitters (indexed by ℓ). The ℓ^{th} transmitter is equipped with $N_{\text{Tx}}^{[\ell]}$ antennas and produces a channel input $\mathbf{X}^{[\ell]} \in \mathbb{R}^{N_{\text{Tx}}^{[\ell]} \times n}$ satisfying an expected power constraint $\mathbb{E}\Big[\mathrm{Tr}\Big(\mathbf{X}^{[\ell]}\mathbf{X}^{[\ell]^{\mathsf{T}}}\Big)\Big] \leq n\rho$ where n is the blocklength.

For IFIA in this dissertation, we will focus on linear encoding strategies: each transmitter has a single codeword $\mathbf{s}^{[\ell]} \in \mathbb{R}^n$ drawn from a codebook of rate $R^{[\ell]}$ that it maps onto its transmit antennas using a beamforming vector $\mathbf{v}^{[\ell]} \in \mathbb{R}^{N_{\text{Tx}}^{[\ell]}}$, $\mathbf{X}^{[\ell]} = \mathbf{v}^{[\ell]}\mathbf{s}^{[\ell]^{\mathsf{T}}}$.

Receivers: There are K receivers (indexed by k). The k^{th} receiver is equipped with $N_{\text{Rx}}^{[k]}$ antennas and observes

$$\mathbf{Y}^{[k]} = \sum_{\ell=1}^{K} \mathbf{H}^{[k,\ell]}\mathbf{X}^{[\ell]} + \mathbf{Z}^{[k]} \tag{7.1}$$

where $\mathbf{H}^{[k,\ell]} \in \mathbb{R}^{N_{\text{Rx}}^{[k]} \times N_{\text{Tx}}^{[\ell]}}$ is the channel matrix from the ℓ^{th} transmitter and $\mathbf{Z}^{[k]}$ is elementwise i.i.d. $\mathcal{N}(0,1)$. The receiver applies an equalization matrix $\mathbf{U}^{[k]} \in \mathbb{R}^{N_{\text{Rx}}^{[k]} \times M^{[k]}}$ (where $M^{[k]}$ will be specified later) to obtain an effective channel output

$$\tilde{\mathbf{Y}}^{[k]} = \mathbf{U}^{[k]^{\mathsf{T}}}\mathbf{Y}^{[k]} \ , \tag{7.2}$$

which is then used to obtain an estimate $\hat{\mathbf{s}}^{[k]}$ of the transmitted codeword $\mathbf{s}^{[k]}$. If the probability of error vanishes with the blocklength, the rates $R^{[1]}, \ldots, R^{[K]}$ are said to be achievable.

We will denote matrices by boldface uppercase symbols (e.g., $\mathbf{H}$) and column vectors by boldface lowercase symbols (e.g., $\mathbf{v}$). We use $^{\mathsf{T}}$ to denote the tanspose and $\mathrm{Tr}(\mathbf{X})$ to represent the trace of a matrix $\mathbf{X}$. Let $\mathbf{A}_{\sim k}$ be the matrix resulting from dropping the k^{th} column of matrix $\mathbf{A}$ and $\mathbf{a}_{\sim k}$ be the vector resulting from dropping the k^{th} entry of vector $\mathbf{a}$. Also, we will use $\|\mathbf{a}\|$ to represent ℓ_2-norm of vector $\mathbf{a}$ and $\log^+(x) \triangleq \max(0, \log(x))$.

Some notation is redefined in this chapter, and should not be confused with notational conventions from earlier.

7.3 Conventional Approach: MaxSINR Algorithm for Zero-Forcing Interference Alignment

We now briefly summarize the key steps of the Max-SINR algorithm [Gomadam et al., 2011], which will act as an inspiration for our IFIA iterative optimization algorithms. Since each transmitter only emits a single data stream, each receiver only needs one equalization vector, $M^{[k]} = 1$. We can express the effective channel output (7.2) as

$$\tilde{\mathbf{Y}}^{[k]} = \underbrace{\mathbf{u}^{[k]\dagger}\mathbf{H}^{[k,k]}\mathbf{v}^{[k]}\mathbf{s}^{[k]\dagger}}_{\text{desired signal}} + \sum_{\ell \neq k} \underbrace{\mathbf{u}^{[k]\dagger}\mathbf{H}^{[k,\ell]}\mathbf{v}^{[\ell]}\mathbf{s}^{[\ell]\dagger}}_{\text{interference}} + \mathbf{u}^{[k]\dagger}\mathbf{Z}^{[k]}$$

where $\mathbf{u}^{[k]} \in \mathbb{R}^{N_{\text{Rx}}^{[k]}}$ is the equalization vector. We assume that each data stream $\mathbf{s}^{[\ell]}$ is drawn from an i.i.d. Gaussian codebook of power ρ and thus each beamforming vector must satisfy $\|\mathbf{v}^{[\ell]}\|^2 \leq 1$. The resulting SINR at receiver k is

$$\text{SINR}_k = \rho \frac{\mathbf{u}^{[k]\dagger}\mathbf{H}^{[k,k]}\mathbf{v}^{[k]}\mathbf{v}^{[k]\dagger}\mathbf{H}^{[k,k]\dagger}\mathbf{u}^{[k]}}{\mathbf{u}^{[k]\dagger}\Big(\sum\limits_{\ell \neq k} \mathbf{H}^{[k,\ell]}\mathbf{v}^{[\ell]}\mathbf{v}^{[\ell]\dagger}\mathbf{H}^{[k,\ell]\dagger} + \mathbf{I}\Big)\mathbf{u}^{[k]}}\ , \tag{7.3}$$

which leads to achievable rates $R^{[k]} = \frac{1}{2}\log(1+\text{SINR}_k)$ since the receiver simply treats the remaining interference as noise.

It can be shown that simultaneously choosing the $\mathbf{u}^{[k]}$ and $\mathbf{v}^{[\ell]}$ to maximize SINR_k for all k is a non-convex optimization problem. However, for a fixed choice of the beamforming vectors $\mathbf{v}^{[\ell]}$, the optimal unit-norm[1] equalization vectors can be expressed in closed form:

$$\mathbf{u}^{[k]} = \frac{\Big(\sum\limits_{\ell \neq k} \mathbf{H}^{[k,\ell]}\mathbf{v}^{[\ell]}\mathbf{v}^{[\ell]\dagger}\mathbf{H}^{[k,\ell]\dagger}\Big)^{-1}\mathbf{H}^{[k,k]}\mathbf{v}^{[k]}}{\Big\|\Big(\sum\limits_{l \neq k} \mathbf{H}^{[k,\ell]}\mathbf{v}^{[\ell]}\mathbf{v}^{[\ell]\dagger}\mathbf{H}^{[k,\ell]\dagger}\Big)^{-1}\mathbf{H}^{[k,k]}\mathbf{v}^{[k]}\Big\|^2}\ . \tag{7.4}$$

Now consider a hypothetical dual channel where the roles of the transmitters

[1] We employ unit-norm equalization and beamforming vectors to make it easier to switch the roles of transmitters and receivers in the dual channel.

and receivers are reversed. Specifically, let $\overleftarrow{\mathbf{H}}^{[k,\ell]} = \mathbf{H}^{[\ell,k]\mathsf{T}}$ denote the dual channel matrix from the ℓ^{th} transmitter to the k^{th} receiver and let $\mathbf{u}^{[\ell]}$ and $\mathbf{v}^{[k]}$ play the role of beamforming and equalization vectors, respectively. It follows that for a fixed choice of the $\mathbf{u}^{[\ell]}$, the optimal $\mathbf{v}^{[k]}$ can be expressed in closed form. Using this fact, it immediately follows that in the original channel, for fixed equalization vectors $\mathbf{u}^{[k]}$, the optimal unit-norm beamforming vectors are

$$\mathbf{v}^{[k]} = \frac{\left(\sum\limits_{\ell \neq k} \overleftarrow{\mathbf{H}}^{[k,\ell]}\mathbf{u}^{[\ell]}\mathbf{u}^{[\ell]\dagger}\overleftarrow{\mathbf{H}}^{[k,\ell]\dagger}\right)^{-1}\overleftarrow{\mathbf{H}}^{[k,k]}\mathbf{u}^{[k]}}{\left\|\left(\sum\limits_{\ell \neq k} \overleftarrow{\mathbf{H}}^{[k,\ell]}\mathbf{u}^{[\ell]}\mathbf{u}^{[\ell]\dagger}\overleftarrow{\mathbf{H}}^{[k,\ell]\dagger}\right)^{-1}\overleftarrow{\mathbf{H}}^{[k,k]}\mathbf{u}^{[k]}\right\|^2}\ . \tag{7.5}$$

Overall, the Max-SINR algorithm uses (7.4) and (7.5) to iteratively optimize the beamforming and equalization vectors. For more details, we refer readers to [Gomadam et al., 2011].

7.4 Integer-Forcing Interference Alignment

In this section, we give a high-level overview of the IFIA strategy, which builds on previous results for compute-and-forward and integer-forcing from [Nazer and Gastpar, 2011, Zhan et al., 2014, Ordentlich et al., 2014, Ntranos et al., 2013b, Nazer et al., 2016]. Our framework inherits the alignment idea from [Ntranos et al., 2013b] as well as the asymmetric compute-and-forward technique from [Nazer et al., 2016]. For details about asymmetric compute-and-forward as well as the complete version for expanded compute-and-forward, we refer readers to [Nazer et al., 2016].

7.4.1 Achievable Rates

Let us assume that the ℓ^{th} transmitter selects a (dithered) lattice codeword $\mathbf{s}_\ell \in \mathbb{R}^n$ with power $\rho_\ell = \frac{1}{n}\mathbb{E}\|\mathbf{s}^{[\ell]}\|^2$ and a beamforming vector $\mathbf{v}^{[\ell]}$ that meets the overall power constraint $\rho_\ell\|\mathbf{v}^{[\ell]}\|^2 = \rho$. Let $\mathbf{P} = \text{diag}(\rho_1, \ldots, \rho_K)$ be the diagonal matrix of coding powers and $\mathbf{S} \triangleq [\mathbf{s}^{[1]} \;\cdots\; \mathbf{s}^{[K]}]^\mathsf{T}$ denote the matrix of codewords. We define the beamforming matrix $\mathbf{V}$ as

$$\mathbf{V} \triangleq \begin{bmatrix} \mathbf{v}^{[1]} & \cdots & \mathbf{0}_{N_{\text{Tx}}^{[1]}} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{N_{\text{Tx}}^{[K]}} & \cdots & \mathbf{v}^{[K]} \end{bmatrix} \tag{7.6}$$

where $\mathbf{0}_N$ refers to the zero column vector of length N. Recall that $\mathbf{H}^{[k,\ell]}$ is the channel matrix from the ℓ^{th} transmitter to the k^{th} receiver. By defining

$$\mathbf{H}^{[k]} \triangleq [\mathbf{H}^{[k,1]} \;\cdots\; \mathbf{H}^{[k,K]}] \;, \tag{7.7}$$

we can compactly write the k^{th} receiver's observation as

$$\mathbf{Y}^{[k]} = \mathbf{H}^{[k]}\mathbf{V}\mathbf{S} + \mathbf{Z}^{[k]} \;. \tag{7.8}$$

The k^{th} receiver's goal is to recover $M^{[k]}$ integer-linear combinations of codewords which can be solved later for the desired codeword $\mathbf{s}^{[k]}$. The i^{th} combination is given by

$$\mathbf{r}_m^{[k]^\mathsf{T}} = \mathbf{a}_m^{[k]^\mathsf{T}}\mathbf{S}, \;\; i = 1, \ldots, M^{[k]} \tag{7.9}$$

where $\mathbf{a}_m^{[k]} \in \mathbb{Z}^K$ is the integer vector containing the integer coefficients of the i^{th} linear combination.

To recover the integer combination for a given $\mathbf{a}_m^{[k]}$, the k^{th} receiver applies the equalization vector $\mathbf{u}_m^{[k]}$ and obtains effective channel output

$$\begin{aligned} \tilde{\mathbf{y}}_m^{[k]^\mathsf{T}} &= \mathbf{u}_m^{[k]^\mathsf{T}}\mathbf{Y}^{[k]} && (7.10) \\ &= \underbrace{\mathbf{r}_m^{[k]^\mathsf{T}}}_{\text{Desired combination}} + \underbrace{\mathbf{z}_{\text{eff},m}^{[k]^\mathsf{T}}}_{\text{Effective noise}} && (7.11) \end{aligned}$$

where $\mathbf{z}_{\text{eff},m}^{[k]^\mathsf{T}} = (\mathbf{u}_m^{[k]^\mathsf{T}}\mathbf{H}^{[k]}\mathbf{V} - \mathbf{a}_m^{[k]^\mathsf{T}})\mathbf{S} + \mathbf{u}_m^{[k]^\mathsf{T}}\mathbf{Z}^{[k]}$ is the effective noise due to the mismatch between the actual effective channel $\mathbf{u}_m^{[k]^\intercal}\mathbf{H}^{[k]}\mathbf{V}$ and the desired integer vector $\mathbf{a}_m^{[k]^\intercal}$ plus the amplified channel noise. The power of the effective noise $\mathbf{z}_{\text{eff},m}^{[k]}$ is given by

$$\left(\sigma_{\text{eff},m}^{[k]}\right)^2 = \frac{1}{n}\mathbb{E}\left[\|\mathbf{z}_{\text{eff},m}^{[k]}\|^2\right] = \|(\mathbf{u}_m^{[k]^\intercal}\mathbf{H}^{[k]}\mathbf{V} - \mathbf{a}_m^{[k]^\intercal})\mathbf{P}^{\frac{1}{2}}\|^2 + \|\mathbf{u}_m^{[k]}\|^2. \tag{7.12}$$

It has been shown in [Nazer et al., 2016] that for a given equivalent channel $\mathbf{H}^{[k]}\mathbf{V}$ and integer matrix $\mathbf{a}_m^{[k]}$, the optimal equalization vector $\mathbf{u}_{\text{opt},m}^{[k]}$ is given by the MMSE equalizer

$$\mathbf{u}_{\text{opt},m}^{[k]^\intercal} = \mathbf{a}_m^{[k]^\intercal}\mathbf{P}^\intercal\mathbf{V}^\intercal\mathbf{H}^{[k]^\intercal}(\mathbf{I} + \mathbf{H}^{[k]}\mathbf{V}\mathbf{P}\mathbf{V}^\intercal\mathbf{H}^{[k]^\intercal})^{-1}. \tag{7.13}$$

Substituting with (7.13) and using Woodbury's matrix identity, we can rewrite (7.12) as

$$\left(\sigma_{\text{eff},m}^{[k]}\right)^2 \triangleq \left\|\mathbf{F}^{[k]}\mathbf{a}_m^{[k]}\right\|^2 \tag{7.14}$$

where $\mathbf{F}^{[k]} = \left(\mathbf{P}^{-1} + \mathbf{V}^\intercal\mathbf{H}^{[k]^\intercal}\mathbf{H}^{[k]}\mathbf{V}\right)^{-\frac{1}{2}}$. We will let $\mathbf{A}^{[k]} \triangleq [\mathbf{a}_1^{[k]} \ \cdots \ \mathbf{a}_{M^{[k]}}^{[k]}]^\intercal$ and $\mathbf{U}^{[k]^\intercal} \triangleq [\mathbf{u}_1^{[k]} \ \cdots \ \mathbf{u}_{M^{[k]}}^{[k]}]^\intercal$ be the matrix representations for the integer coefficients and equalization.

In order to decode the m^{th} integer-combination at the the k^{th} receiver, all participating users with non-zero coefficient in $\mathbf{a}_m^{[k]^\intercal}$ should design their codebook to tolerant noise power $\left(\sigma_{\text{eff},m}^{[k]}\right)^2$ given in (7.12). Successfully decoding the m^{th} integer-combination at the the k^{th} receiver results in a certain constraint called the *computation rate* for each participating users, given as

$$R_{\text{comp},m,\ell}^{[k]} = \frac{1}{2}\log^+\left(\frac{\rho_\ell}{\left(\sigma_{\text{eff},m}^{[k]}\right)^2}\right) \quad \text{for} \quad a_{m,\ell}^{[k]^\intercal} \neq 0 \tag{7.15}$$

where $a_{m,\ell}^{[k]^\intercal}$ is the ℓ^{th} entry of $\mathbf{a}_m^{[k]^\intercal}$. Here $R_{\text{comp},m,\ell}^{[k]}$ is a rate constraint to user ℓ only if this user participates in the m^{th} integer-combination at the the k^{th} receiver. The message from the ℓ^{th} transmitter (user) might participate in multiple combinations at multiple receivers. Thus the achievable rate for the ℓ^{th} transmitter is mapped from one of the computation rates given as

$$R^{[\ell]} = \min_{k=1,\ldots,K} \min_{m: a_{m,\ell}^{[k]\mathsf{T}} \neq 0} R_{\text{comp},m,\ell}^{[k]} \tag{7.16}$$

For asymmetric compute-and-forward, the achievable rate for the ℓ^{th} user depends on a valid pairing relationship (k, m, ℓ) between its coding power ρ_ℓ and one of the effective noise power $\left(\sigma_{\text{eff},m}^{[k]}\right)^2$. The rate expression in (7.16) can be further improved by implementing *algebraic successive cancellation* introduced in [Ordentlich et al., 2014] which relaxes the pairing constraints. We will first review the basic idea of algebraic successive cancellation and then show the improved achievable rates region.

Algebraic successive cancellation can be achieved by using previously decoded integer-combinations to eliminate some of the users codewords contributing in the subsequent integer-combinations which relaxes the computation rate constraints on these users. In order to capture the order in which the codewords can be eliminated from the integer-combinations, we define a *mapping* $\mathcal{I}^{[k]}$ as a set of pair of the form (m, ℓ), where $\ell \in \{1, \ldots, K\}$ denotes the user index, $m \in \{1, \ldots, M^{[k]}\}$ denotes the integer-combination index and $(m, \ell) \in \mathcal{I}^{[k]}$ means that the ℓ^{th} user can not be canceled out while decoding the m^{th} integer-combination. Only some mappings are *admissible* (depending on the integer matrix $\mathbf{A}^{[k]}$). A mapping $\mathcal{I}^{[k]}$ is said to be admissible if there exists a lower unitriangular[2] matrix $\mathbf{L}^{[k]} \in \mathbb{R}^{M^{[k]} \times M^{[k]}}$ such that the $(m, \ell)^{\text{th}}$ entry of $\mathbf{L}^{[k]}\mathbf{A}^{[k]}$ is equal to zero for all $(m, \ell) \notin \mathcal{I}^{[k]}$. The admissible mapping $\mathcal{I}^{[k]}$ captures the possible assignments of the computation rates to the users.

For any choice of integer matrices $\mathbf{A}^{[k]}$, beamforming matrix $\mathbf{V}$, equalization ma-

[2]lower triangular matrix whose diagonal elements are equal to 1

trices $\mathbf{U}^{[k]}$, and admissible mappings $\mathcal{I}^{[k]}$, the following rates are achievable

$$R^{[\ell]} = \min_{k=1,\dots,K} \min_{m:(m,\ell)\in\mathcal{I}^{[k]}} R^{[k]}_{\text{comp},m,\ell} \tag{7.17}$$

For an in-depth discussion of the achievability proof, we refer interested readers to [Ordentlich et al., 2014, Ntranos et al., 2013b] and [Nazer et al., 2016].

7.4.2 Integer matrix $\mathbf{A}^{[k]}$ structure

In the simple case where the k^{th} receiver wants to decode all of the codewords (K codewords), we need K independent integer-combinations such that

$$\operatorname{rank}\left(\mathbf{A}^{[k]}\right) = K. \tag{7.18}$$

Decoding all the codewords is essential in the multiple-access channels, where the receiver is interested in all the transmitted codewords. In the interference channel, this overconstrains the user rates as we will have K computation rate constraints while the receiver only desires one message.

The receiver can choose to decode less number of integer-combinations (i.e., $M^{[k]}$ instead of K), then solve for the desired codewords. In order to solve for the desired codewords, the following conditions are needed:

$$\operatorname{rank}\left(\mathbf{A}^{[k]}\right) = M^{[k]}, \tag{7.19}$$

$$\operatorname{rank}\left(\mathbf{A}^{[k]}_{\sim k}\right) = M^{[k]} - 1. \tag{7.20}$$

In other words, the $M^{[k]}$ integer-combinations should be linearly independent and the coefficients of the interference codewords in the $M^{[k]}$ integer-combinations should be aligned in no more than $M^{[k]} - 1$ dimensional space.

Example 1. *Consider the case when $K = 3$ and $M^{[1]} = 2$. An example of the* *integer-combinations that could be decoded at receiver 1 are*

$$\mathbf{r}_1^{[1]} = \mathbf{s}^{[1]} + \mathbf{s}^{[2]} + 2\mathbf{s}^{[3]} \tag{7.21}$$

$$\mathbf{r}_1^{[1]} = -\mathbf{s}^{[1]} + 2(\mathbf{s}^{[2]} + 2\mathbf{s}^{[3]}). \tag{7.22}$$

It can be shown that $\mathbf{s}^{[1]} = \frac{2\mathbf{r}_1^{[1]} - \mathbf{r}_2^{[1]}}{3}$. *The integer matrices* $\mathbf{A}^{[1]}$ *and* $\mathbf{A}_{\sim 1}^{[1]}$ *are*

$$\mathbf{A}^{[1]} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{7.23}$$

$$\mathbf{A}_{\sim 1}^{[1]} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{7.24}$$

which satisfy the conditions in (7.19) *and* (7.20)*.*

7.4.3 Optimization Issues

The natural objective of IFIA is to choose the beamforming matrix $\mathbf{V}$, equalization matrices $\mathbf{U}^{[k]}$ and integer matrices $\mathbf{A}^{[k]}$ jointly to maximize the sum of the achievable rates (user rates). Unfortunately, this is a non-convex problem. Instead of solving the original problem, in Section 7.6 we will propose two suboptimal algorithms to perform IFIA using iterative optimization techniques. In the iterative algorithms, we will split the part of choosing integer matrix $\mathbf{A}^{[k]}$ from the part of the selecting beamforming matrix $\mathbf{V}$ and the equalization matrix $\mathbf{U}^{[k]}$. As a first step, we will discuss how to choose the integer matrix $\mathbf{A}^{[k]}$ in Section 7.5. We then talk about how to choose $\mathbf{V}$ and $\mathbf{U}^{[k]}$ for a fixed integer matrix $\mathbf{A}^{[k]}$ in Section 7.6.

Even for a fixed $\mathbf{A}^{[k]}$, optimizing $\mathbf{U}^{[k]}$ and $\mathbf{V}$ jointly for the sum rate or the symmetric rate is still a non-convex optimization problem. We tackle this problem by further relaxing the optimization problem into an alternative optimization problem for $\mathbf{V}$ and $\mathbf{U}^{[k]}$.

The achievable rates in (7.17) also depend on the set of admissible mappings $\mathcal{I}^{[k]}$. These mappings are determined by the structure of $\mathbf{A}^{[k]}$ and their patterns make the optimization even more complicated. While optimizing $\mathbf{A}^{[k]}$, $\mathbf{U}^{[k]}$ and $\mathbf{V}$, we will not take the admissible mapping issue into our consideration. We will exhaustively search among all admissible mapping at the very end once $\mathbf{A}^{[k]}$, $\mathbf{U}^{[k]}$ and $\mathbf{V}$ are fixed.

For optimization, we will consider the problem of maximizing the sum of computation rates at the k^{th} receiver. According to (A.1), maximizing the sum of computation rates equals to minimizing the multiplication of the effective noise powers $\prod_{i=1}^{M^{[k]}} \|\mathbf{F}^{[k]}\mathbf{a}_m^{[k]}\|^2$. Combining the conditions of combinations mentioned before, this problem is equivalent to find the shortest $M^{[k]}$ vectors in the lattice spanned by $\mathbf{F}^{[k]}$ satisfying (7.19) and (7.20).

Finding the shortest and independent $M^{[k]}$ vectors in a lattice spanned by $\mathbf{F}^{[k]}$ is, in general, a hard problem. Some polynomial time algorithms (e.g., LLL algorithm [Lenstra et al., 1982]) can provide a good approximation, but there is no guarantee for satisfying the conditions in (7.19) and (7.20). In the next section, we propose a new, yet suboptimal, algorithm which holds the conditions we need while approximating to the shortest vectors.

7.5 Aligned Lattice Reduction Algorithms

In this section, we will discuss how to choose the integer-combinations coefficients $\mathbf{A}^{[k]}$ to minimize the noise powers. For simplicity, we will propose an algorithm to optimize the integer matrix $\mathbf{A}^{[k]}$ for a fixed $M^{[k]} = 2$ (decode two integer-combinations) and given matrices $\mathbf{V}$ and $\mathbf{U}^{[k]}$. We name this algorithm Aligned LLL Method-I. Recall that decoding a number of integer combinations $M^{[k]}$ less than the number of unknown codewords K requires the integer matrix $\mathbf{A}^{[k]}$ to have a special structure as mentioned earlier. We will introduce a lattice reduction method called aligned LLL algorithm to obtain the desired $\mathbf{A}^{[k]}$. For the general case ($M^{[k]} \geq 2$), a generalization of our aligned LLL algorithm is given in the Appendix A as well as simulation results.

For $M^{[k]} = 2$ (i.e., $\mathbf{A}^{[k]} \in \mathbb{Z}^{2\times K}$), we need to align $K-1$ interfering codewords into a single combination. Recall that $\mathbf{a}_m^{[k]^\intercal}$ is the m^{th} row of $\mathbf{A}^{[k]}$ for $m = 1, 2$ and $\mathbf{a}_{m\sim k}^{[k]^\intercal}$ is the vector resulting from dropping the k^{th} entry of vector $\mathbf{a}_m^{[k]^\intercal}$. At the k^{th} receiver, the constraints in (7.19) and (7.20) can be rewritten as constraints for $\mathbf{a}_1^{[k]^\intercal}$ and $\mathbf{a}_2^{[k]^\intercal}$ such that

$$\mathbf{a}_{1\sim k}^{[k]} = b_{1,2}^{[k]}\mathbf{a}_{\text{int}}^{[k]}, \qquad \mathbf{a}_{2\sim k}^{[k]} = b_{2,2}^{[k]}\mathbf{a}_{\text{int}}^{[k]} \tag{7.25}$$

$$a_{1,k}^{[k]} = b_{1,1}^{[k]}, \qquad a_{2,k}^{[k]} = b_{2,1}^{[k]} \tag{7.26}$$

$$\text{rank}\left(\begin{bmatrix} b_{1,1}^{[k]} & b_{1,2}^{[k]} \\ b_{2,1}^{[k]} & b_{2,2}^{[k]} \end{bmatrix}\right) = 2 \tag{7.27}$$

where $\mathbf{a}_{\text{int}}^{[k]} \in \mathbb{Z}^{K-1}$ and $b_{m,i}^{[k]} \in \mathbb{Z}, \forall i, m = 1, 2$.

Here the vector $\mathbf{a}_{\text{int}}^{[k]}$ represents the coefficients of an aligned function. Recall that $\mathbf{S} \triangleq [\mathbf{s}^{[1]} \ \cdots \ \mathbf{s}^{[K]}]^\intercal$ denote the matrix of codewords and let $(\mathbf{S}^\intercal)_{\sim k}$ be the matrix resulting from dropping the k^{th} column of $\mathbf{S}^\intercal$. Another way to view this, is to define an aligned function of interfering codewords as

$$\mathbf{g}^{[k]} = (\mathbf{S}^\intercal)_{\sim k}\mathbf{a}_{\text{int}}^{[k]}. \tag{7.28}$$

Now, we decode two independent integer-combinations $\mathbf{r}_1^{[k]}$ and $\mathbf{r}_2^{[k]}$ given by

$$\mathbf{r}_1^{[k]} = b_{1,1}^{[k]}\mathbf{s}^{[k]} + b_{1,2}^{[k]}\mathbf{g}^{[k]} \tag{7.29}$$

$$, \mathbf{r}_2^{[k]} = b_{2,1}^{[k]}\mathbf{s}^{[k]} + b_{2,2}^{[k]}\mathbf{g}^{[k]}. \tag{7.30}$$

If $\mathbf{r}_1^{[k]}$ and $\mathbf{r}_2^{[k]}$ are decoded successfully, we can solve for $\mathbf{s}^{[k]}$ (and $\mathbf{g}^{[k]}$). Our goal is to find the optimal (or a good approximation) $\mathbf{a}_1^{[k]^\intercal}$ and $\mathbf{a}_2^{[k]^\intercal}$ given by the structure in (7.25)-(7.27) to minimize the product of effective noise powers $\prod_{m=1,2} \|\mathbf{F}^{[k]}\mathbf{a}_m^{[k]}\|^2$.

We will propose a method based on Minkowski's Second Theorem [Cassels, 1957]. The method allows us to get a theoretical lower bound on the computation sum rate. For any chosen interference function $\mathbf{g}^{[k]}$ in (7.28), we choose two independent integer vectors $\mathbf{b}_1^{[k]} = [b_{1,1}^{[k]} \; b_{1,2}^{[k]}]^\intercal$ and $\mathbf{b}_2^{[k]} = [b_{2,1}^{[k]} \; b_{2,2}^{[k]}]^\intercal$ to minimize $\prod_{m=1}^{2}(\sigma_{\text{eff},m}^{[k]})^2$ which can be written as

$$\prod_{m=1}^{2}(\sigma_{\text{eff},m}^{[k]})^2 = \prod_{m=1}^{2} \|\mathbf{F}^{[k]}\mathbf{a}_m^{[k]}\|^2 \tag{7.31}$$

$$= \prod_{m=1}^{2} \|b_{m,1}^{[k]}\mathbf{f}_k^{[k]} + b_{m,2}^{[k]}\mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\text{int}}^{[k]}\|^2 \tag{7.32}$$

$$= \prod_{m=1}^{2} \|\mathbf{F}_{\text{red}}^{[k]}\mathbf{b}_m^{[k]}\|^2 \tag{7.33}$$

where $\mathbf{F}_{\text{red}}^{[k]} = \begin{bmatrix}\mathbf{f}_k^{[k]} & \mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\text{int}}^{[k]}\end{bmatrix}$ represents the basis of a new reduced lattice. The first column of this basis corresponds to the desired signal $\mathbf{s}^{[k]}$, while the second column corresponds to the interference function $\mathbf{g}^{[k]}$.

We can choose $\mathbf{b}_1^{[k]}$ and $\mathbf{b}_2^{[k]}$ by finding the two shortest non-zero vectors in this new reduced lattice with basis given by $\mathbf{F}_{\text{red}}^{[k]}$. From (7.33), the optimal $\mathbf{b}_1^{[k]}$ and $\mathbf{b}_2^{[k]}$ are given as a function of $\mathbf{a}_{\text{int}}^{[k]}$ by

$$\mathbf{b}_1^{[k]} = \arg\min_{\mathbf{b}} \|\mathbf{F}_{\text{red}}^{[k]}\mathbf{b}\|^2 \tag{7.34}$$

$$\mathbf{b}_2^{[k]} = \arg\min_{\mathbf{b}:\text{rank}([\mathbf{b}_1^{[k]}\mathbf{b}])=2} \|\mathbf{F}_{\text{red}}^{[k]}\mathbf{b}\|^2. \tag{7.35}$$

Lattice reduction algorithms (e.g., the LLL algorithm) can be used to give approximate solutions. To apply Minkowski's Second Theorem, recall the definition of successive minimum in Chapter 6 Definition 5. From [Feng et al., 2013] and (7.33), the powers of the effective noise in both integer-combinations are given by the first and second successive minima of the reduced lattice $\mathbf{F}_{\text{red}}^{[k]}$ such that $(\sigma_{\text{eff},1}^{[k]})^2 = \lambda_1^2(\mathbf{F}_{\text{red}}^{[k]})$ and $(\sigma_{\text{eff},2}^{[k]})^2 = \lambda_2^2(\mathbf{F}_{\text{red}}^{[k]})$. We can write the sum of the computation rates as

$$\sum_{m=1}^{2} R_{\mathrm{comp},m} = \frac{1}{2}\sum_{m=1}^{2}\log\left(\frac{\rho}{(\sigma_{\mathrm{eff},m}^{[k]})^2}\right) \tag{7.36}$$

$$= \frac{1}{2}\log\left(\frac{\rho^2}{\prod_{m=1}^{2}\lambda_m^2(\mathbf{F}_{\mathrm{red}}^{[k]})}\right) \tag{7.37}$$

$$\overset{a}{\geq} \frac{1}{2}\log\left(\frac{\rho^2}{4|\det(\mathbf{F}_{\mathrm{red}}^{[k]\mathsf{T}}\mathbf{F}_{\mathrm{red}}^{[k]})|}\right) \tag{7.38}$$

where a is due to Minkowski's Second Theorem [Cassels, 1957]. Furthermore, we can write $\det(\mathbf{F}_{\mathrm{red}}^{[k]\mathsf{T}}\mathbf{F}_{\mathrm{red}}^{[k]})$ as

$$\det(\mathbf{F}_{\mathrm{red}}^{[k]\mathsf{T}}\mathbf{F}_{\mathrm{red}}^{[k]}) = \|\mathbf{f}_k^{[k]}\|^2\|\mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\mathrm{int}}^{[k]}\|^2 - (\mathbf{f}_k^{[k]\mathsf{T}}\mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\mathrm{int}}^{[k]})^2 \tag{7.39}$$

$$= \mathbf{a}_{\mathrm{int}}^{[k]\mathsf{T}}\mathbf{F}_{\sim k}^{[k]\mathsf{T}}\|\mathbf{f}_k^{[k]}\|\left(\mathbf{I} - \frac{\mathbf{f}_k^{[k]}\mathbf{f}_k^{[k]\mathsf{T}}}{\|\mathbf{f}_k^{[k]}\|^2}\right)\|\mathbf{f}_k^{[k]}\|\mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\mathrm{int}}^{[k]} \tag{7.40}$$

$$= \|\mathbf{G}^{[k]}\mathbf{a}_{\mathrm{int}}^{[k]}\|^2 \tag{7.41}$$

where $\mathbf{G}^{[k]}$ can be obtained by Cholesky factorization such that

$$\mathbf{G}^{[k]}\mathbf{G}^{[k]\mathsf{T}} = (\mathbf{F}_{\sim k}^{[k]\mathsf{T}}\|\mathbf{f}_k^{[k]}\|\left(\mathbf{I} - \frac{\mathbf{f}_k^{[k]}\mathbf{f}_k^{[k]\mathsf{T}}}{\|\mathbf{f}_k^{[k]}\|^2}\right)\|\mathbf{f}_k^{[k]}\|\mathbf{F}_{\sim k}^{[k]}) \tag{7.42}$$

. Finally, the interference function $\mathbf{g}^{[k]}$ (i.e., $\mathbf{a}_{\mathrm{int}}^{[k]}$) can be obtained by lattice reduction on $\mathbf{G}^{[k]}$:

$$\mathbf{a}_{\mathrm{int}}^{[k]} = \arg\min_{\mathbf{a}\in\mathbb{Z}^{K-1}}\|\mathbf{G}^{[k]}\mathbf{a}\|^2\ . \tag{7.43}$$

Choosing $\mathbf{a}_{\mathrm{int}}^{[k]}$ as in (7.43) guarantees that $\|\mathbf{G}^{[k]}\mathbf{a}_{\mathrm{int}}^{[k]}\|^2$ will be the shortest vector in a lattice with basis $\mathbf{G}^{[k]}$ (i.e., $\lambda_1^2(\mathbf{G}^{[k]})$) and as a result we can bound the sum of the computation rates as

$$\sum_{m=1}^{2} R_{\text{comp},m} \geq \frac{1}{2}\log\left(\frac{\mathsf{SNR}^2}{4\lambda_1^2(\mathbf{G}^{[k]})}\right) \tag{7.44}$$

$$\overset{a}{\geq} \frac{1}{2}\log\left(\frac{\mathsf{SNR}^2}{4(K-1)\det(\mathbf{G}^{[k]})^{\frac{2}{K-1}}}\right) \tag{7.45}$$

where a is due to Minkowski's first theorem. Algorithm 3 shows the details of Method-I.

Algorithm 3 Aligned LLL algorithm for decoding 2 integer-combinations (Method-I)

1. Step1: Using the LLL algorithm, find the shortest vector in the lattice $\mathbf{F}^{[k]}$

$$\mathbf{a}_{\text{int}}^{[k]} = \arg\min_{\mathbf{a}\in\mathbb{Z}^{K-1}} \|\mathbf{G}^{[k]}\mathbf{a}\|^2$$

   where $\mathbf{G}^{[k]}$ can be obtained by factoring $\mathbf{G}^{[k]\mathsf{T}}\mathbf{G}^{[k]} = \mathbf{F}_{\sim k}^{[k]\mathsf{T}}\|\mathbf{f}_k^{[k]}\|\left(\mathbf{I}-\frac{\mathbf{f}_k^{[k]}\mathbf{f}_k^{[k]\mathsf{T}}}{\|\mathbf{f}_k^{[k]}\|^2}\right)\|\mathbf{f}_k^{[k]}\|\mathbf{F}_{\sim k}^{[k]}$ using Cholesky decomposition.

2. Step2: Using the LLL algorithm, find the shortest two vectors in the lattice $\mathbf{F}_{\text{red}}{}^{[k]} = \begin{bmatrix}\mathbf{f}_k^{[k]} & \mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{\text{int}}^{[k]}\end{bmatrix}$

$$\mathbf{b}_i^{[k]} = \arg\min_{\mathbf{b}\in\mathbb{Z}^2:\text{rank}([\mathbf{b}_1^{[k]},\ldots,\mathbf{b}_{i-1}^{[k]},\mathbf{b}])=i} \|\mathbf{F}_{\text{red}}{}^{[k]}\mathbf{b}\|^2,\ i=1,2$$

3. Step3: Calculate the integer matrix $\mathbf{A}^{[k]}$ using

$$\tilde{\mathbf{A}}^{[k]} = \begin{bmatrix} b_{1,1}^{[k]} & b_{1,2}^{[k]}\mathbf{a}_{\text{int}}^{[k]} \\ b_{2,1}^{[k]} & b_{2,2}^{[k]}\mathbf{a}_{\text{int}}^{[k]} \end{bmatrix}$$
$$\mathbf{A}^{[k]} = \mathbf{L}^{[k]}(\tilde{\mathbf{A}}^{[k]})$$

   where $\mathbf{L}^{[k]}$ is a permutation matrix which put column 1 in between columns k and $k+1$ of $\tilde{\mathbf{A}}^{[k]}$.

7.6 IFIA through CVX and Duality

In this section, we propose two methods to optimize beamforming matrix $\mathbf{V}$ and equalization matrix $\mathbf{U}^{[k]}$. We assume the integer combination coefficients ($\mathbf{A}^{[k]}$) is fixed while optimizing $\mathbf{V}$ and $\mathbf{U}^{[k]}$. As mentioned before, jointly optimizing $\mathbf{V}$ and $\mathbf{U}^{[k]}$ is not a convex problem, and we will relax the joint optimization problem into two separate optimization problems.

For any given beamforming matrix, $\mathbf{V}$, the optimal equalization matrix $\mathbf{U}^{[k]}$ is always in the form of MMSE equalizer as in (7.13). For a given equalization matrix $\mathbf{U}^{[k]}$, we develop two algorithms to update the beamforming matrix $\mathbf{V}$. The first algorithm relaxes the problem of choosing $\mathbf{V}$, given $\mathbf{A}^{[k]}$ and $\mathbf{U}^{[k]}$, to a convex optimization problem. We can use convex optimization toolbox, like the CVX package [Grant et al., 2008], to solve the relaxed convex problem. The second algorithm iteratively optimizes $\mathbf{V}$ and $\mathbf{U}^{[k]}$ using the idea of channel reciprocity and uplink-downlink duality for integer-forcing [He et al., 2014]. Both algorithms are iterative optimization algorithms. For each iteration, the integer matrix $\mathbf{A}^{[k]}$ can be updated using the aligned LLL algorithm in Section 7.5 for fixed beamforming matrix $\mathbf{V}$ and equalization matrix $\mathbf{U}^{[k]}$.

We will use the CVX package to solve the convex optimization problem for our first algorithm, thus, we name the first algorithm CVX-IFIA. The second algorithm will be called Dual-IFIA since it borrows the idea of duality for integer-forcing from [He et al., 2014].

7.6.1 CVX-IFIA

The achievable rates of the IFIA are bounded by the computation rates. Instead of optimizing the sum of the achievable rates, we will relax the original problem to optimize the worst computation rate[3] (i.e., largest effective noise power) across all the receivers. The relaxed problem can be written as

$$\begin{aligned} \mathbb{P}1: \quad & \min_{\mathbf{V},\mathbf{U}^{[k]}} \left(\max_{i,k} \left(\sigma_{\text{eff},i}^{[k]} \right)^2 \right) \\ & s.t \ \ \|\mathbf{v}^{[\ell]}\|^2 \leq 1, \ \ \forall \ell. \end{aligned} \tag{7.46}$$

Here, $\mathbb{P}1$ is still a non-convex optimization problem. However, for a fixed $\mathbf{U}^{[k]}$, $\mathbb{P}1$ can be rewritten as

$$\begin{aligned} \mathbb{P}2: \quad & \min_{\mathbf{V}} \left(\max_{i,k} \left(\sigma_{\text{eff},i}^{[k]} \right)^2 \right) \\ & s.t \ \ \|\mathbf{v}^{[\ell]}\|^2 \leq 1, \ \ \forall \ell. \end{aligned} \tag{7.47}$$

which is a convex optimization problem. Since for any fixed beamforming matrix $\mathbf{V}$, (7.13) gives the optimal $\mathbf{U}^{[k]}$, one can iteratively optimize $\mathbf{U}^{[k]}$ and $\mathbf{V}$ using (7.13) and the solution of (7.47). The details of the algorithm is presented in Algorithm 2. To guarantee better performance, the CVX-IFIA algorithm is initialized by the beamforming vectors given by the Max-SINR algorithm described in Section 7.3.

7.6.2 Dual-IFIA

Before giving the details of the algorithm, we introduce the dual channel and dual network for the IFIA. In the primal network, each receiver k wants to decode $M^{[k]}$ combinations and solve for the desired single codeword $\mathbf{s}^{[k]}$ sent by the k^{th} transmitter. Overall, we have $M = \sum_k M^{[k]}$ combinations decoded at all the receivers.

In the dual network, the receivers and transmitters roles are reversed. The ℓ^{th} primal receiver becomes the ℓ^{th} dual transmitter and the k^{th} primal transmitter becomes the k^{th} dual receiver (i.e., the dual channel matrix $\overleftarrow{\mathbf{H}}^{[k,\ell]} = \mathbf{H}^{[\ell,k]^{\intercal}}$). In addition, the beamforming (equalization) vectors of the primal network become the equalization (beamforming) vectors of the dual network. As a result, in the dual network,

[3] The intuition here was to maximize the symmetric rate.

each dual transmitter ℓ wants to send $M^{[\ell]}$ messages, while each dual receiver k wants to decode only one combination of these messages. We have dual beamforming matrices $\overleftarrow{\mathbf{V}}^{[\ell]} \in \mathbb{R}^{N_{\text{Rx}}^{[\ell]} \times M^{[\ell]}}$ and the dual equalization vectors $\overleftarrow{\mathbf{u}}^{[k]} \in \mathbb{R}^{N_{\text{Tx}}^{[k]}}$. Let $\mathbf{A} = [\mathbf{A}^{[1]^\intercal} \ \cdots \ \mathbf{A}^{[K]^\intercal}]^\intercal \in \mathbb{Z}^{M \times K}$ be the integer matrix of the primal channel, the dual integer matrix can be represented as

$$\overleftarrow{\mathbf{A}} = \mathbf{A}^\intercal \in \mathbb{Z}^{K \times M} \tag{7.48}$$

where K here represents the total number of combinations and M represents the total number of the transmitted messages. Define the channel to the k^{th} dual receiver as

$$\overleftarrow{\mathbf{H}}^{[k]} = \left[\overleftarrow{\mathbf{H}}^{[k,1]} \ \cdots \ \overleftarrow{\mathbf{H}}^{[k,K]} \right]. \tag{7.49}$$

Following the same steps as in the primal IFIA, the k^{th} dual receiver decodes a single combination and we can write the power of the effective noise as

$$\left(\overleftarrow{\sigma}_{\text{eff}}^{[k]}\right)^2 \triangleq \|\overleftarrow{\mathbf{u}}^{[k]^\intercal}\|^2 + \left(\overleftarrow{\mathbf{u}}^{[k]^\intercal} \overleftarrow{\mathbf{H}}^{[k]} \overleftarrow{\mathbf{V}} - \overleftarrow{\mathbf{a}}^{[k]^\intercal}\right) \overleftarrow{\mathbf{P}} \left(\overleftarrow{\mathbf{u}}^{[k]^\intercal} \overleftarrow{\mathbf{H}}^{[k]} \overleftarrow{\mathbf{V}} - \overleftarrow{\mathbf{a}}^{[k]^\intercal}\right)^\intercal$$

where $\overleftarrow{\mathbf{V}}$ is the block diagonal matrix of $[\overleftarrow{\mathbf{V}}^{[1]} \ \cdots \ \overleftarrow{\mathbf{V}}^{[K]}]$ and $\overleftarrow{\mathbf{P}}$ is the diagonal coding power matrix with diagonal elements

$$\overleftarrow{P}_{i,i} = \frac{\rho}{\|\overleftarrow{\mathbf{v}}_i\|^2} \tag{7.50}$$

The optimal equalization vector $\overleftarrow{\mathbf{u}}^{[k]^\intercal}$ which minimizes the effective noise power $\overleftarrow{\sigma}_{\text{eff}}^{[k]}$ at the k^{th} dual receiver is

$$\overleftarrow{\mathbf{u}}_{\text{opt}}^{[k]^\intercal} = \overleftarrow{\mathbf{A}}^{[k]} \overleftarrow{\mathbf{P}}^\intercal \overleftarrow{\mathbf{V}}^\intercal \overleftarrow{\mathbf{H}}^{[k]^\intercal} (\mathbf{I} + \overleftarrow{\mathbf{H}}^{[k]} \overleftarrow{\mathbf{V}} \overleftarrow{\mathbf{P}} \overleftarrow{\mathbf{V}}^\intercal \overleftarrow{\mathbf{H}}^{[k]^\intercal})^{-1}. \tag{7.51}$$

We can use the equalization vectors $\overleftarrow{\mathbf{u}}^{[k]}$ at the k^{th} dual receiver (after normalizing) and then map it to the beamforming vectors $\mathbf{v}^{[k]}$ for the k^{th} primal transmitter.

We can iteratively use the closed form expressions in (7.13) and (7.51) to optimize the beamforming and equalization vectors. The details of the the proposed algorithm is given in Algorithm. 4.

7.7 Numerical Results

We now briefly investigate the performance of our iterative algorithms. In our simulations, we consider the case of 500 channel realizations. Recall that for symmetric systems, there is a feasibility condition for the existence of a linear strategy for interference alignment in a DoF sense [Yetis et al., 2010]. The condition is given by $N_{\text{Tx}} + N_{\text{Rx}} - (K+1)d \geq 0$ where N_{Tx} is the number of antennas for each transmitter, N_{Rx} is the number of antennas for each receiver, K is the number of users and d is the DoF demanded by each user. In this dissertation, the DoF demanded by each user is 1.

Figure 7·2 shows the sum rate of three users obtained after 20 iterations. To generate the plot, we have set $K = 3$, each transmitter and receiver have two antennas and $M^{[k]} = 2, \forall k$. Notice that this system setting satisfies the feasible scenario in [Yetis et al., 2010]. The elements of the channel matrices $\mathbf{H}^{[k,\ell]}$ are drawn i.i.d. $\mathcal{N}(0,1)$. Notice that the Max-SINR algorithm is a special case of IFIA by setting integer matrix $\mathbf{A}$ to be identity. Thus, we can use Max-SINR algorithm without changing our decoding framework. In Figure 7·2, "max all" represents the maximum rate achieved among IFIA(Duality), IFIA(CVX) and Max-SINR algorithm. "Decode All" is the scenario where $M^{[k]} = 3$ and there is no interference alignment. For clarity, we have omitted the plot of max(Max-SINR, Dual-IFIA) as well as max(Max-SINR, CVX-IFIA) since they are very close to "max all". From Figure 7·2, it can be observed that IFIA(both Dual and CVX variations) can achieve the same degrees-of-freedom as the Max-SINR algorithm. The performance can be ordered from the highest sum rate

Algorithm 4 Dual/CVX-IFIA Iterative Optimization

Given Iteration Number, power constraint ρ and $\mathbf{H}^{[k,\ell]}, \forall k, \ell$.

1. Initialization: counter=0, $\mathbf{v}^{[k]}$, $\mathbf{U}^{[k]}$, $\rho_k = \rho$, $\forall k$.
2. Run Max SINR algorithm and update $\mathbf{v}^{[k]}$ and $\mathbf{U}^{[k]}$, $\forall k$.
3. Choose $\mathbf{A}$ using Algorithm 1.
4. Optimize $\mathbf{U}^{[k]}$ using (7.13), $\forall k$.
5. **while** counter $<$ Iteration Number **do**
   (a) **if** Dual-IFIA
      i. Set $\overleftarrow{\mathbf{A}} = \mathbf{A}^{\intercal}$, $\overleftarrow{\mathbf{H}}^{[k,\ell]} = \mathbf{H}^{[\ell,k]\intercal}$ and $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}$.
      ii. Optimize $\overleftarrow{\mathbf{u}}^{[k]\intercal}$ using (7.51), $\forall k$.
      iii. **if** $||\overleftarrow{\mathbf{u}}^{[k]}||^2 > 1$ **then**
         Normalize $\overleftarrow{\mathbf{u}}^{[k]}$.
      iv. **end if**
      v. Set $\mathbf{v}^{[k]} = \overleftarrow{\mathbf{u}}^{[k]}$, $\forall k$.

   else if CVX-IFIA
      i. Using CVX package, solve $\mathbb{P}1$
      ii. Optimize $\mathbf{U}^{[k]}$ using (7.13), $\forall k$.

   end if
   (b) Update $\rho_k = \rho/||\mathbf{v}^{[k]}||^2$, $\forall k$.
   (c) Update $\mathbf{A}$ using Algorithm 1.
   (d) counter=counter+1.
6. **end while**
7. Optimize $\mathbf{U}^{[k]}$ using (7.13), $\forall k$.
8. Output ρ_k, $\mathbf{A}^{[k]}$, $\mathbf{v}^{[k]}$ and $\mathbf{U}^{[k]}$, $\forall k$.

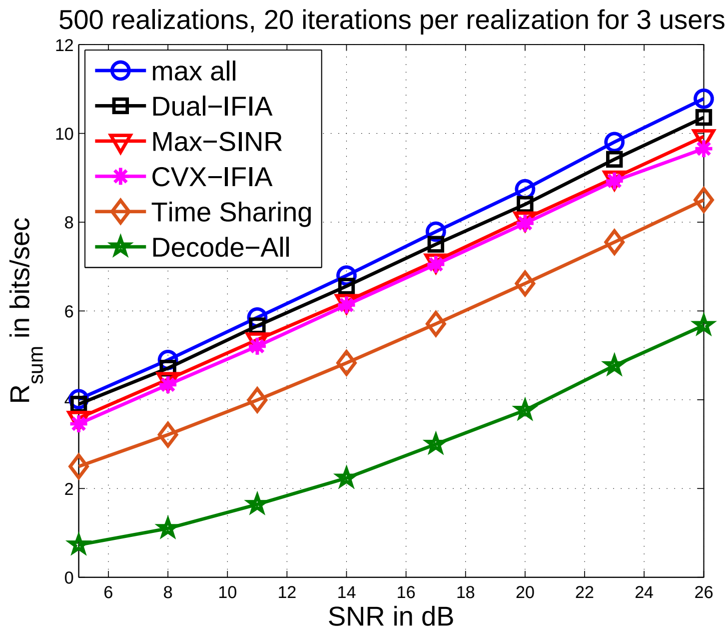

Figure 7·2: Performance of iterative algorithms for IFIA in feasible scenario: $K = 3$, $N_{\text{Tx}}^{[k]} = N_{\text{Rx}}^{[k]} = 2$, $M^{[k]} = 2, \forall k$.

to the lowest in the following order: "max all", Dual-IFIA, Max-SINR, CVX-IFIA, "Time Sharing" and "Decode-All"

Figure 7·3 shows the sum rates for 4 users ($K = 4$), each transmitter and receiver has two antennas and $M^{[k]} = 2, \forall k$. This system setting does not satisfy the feasible scenario in [Yetis et al., 2010]. For clarity we ignore the plot of max(Max-SINR, CVX-IFIA) and max(Max-SINR, Dual-IFIA), they are only a little bit higher than Dual-IFIA. It can be seen that Max-SINR algorithm degrades significantly and IFIA outperforms Max-SINR, especially when SNR is high.

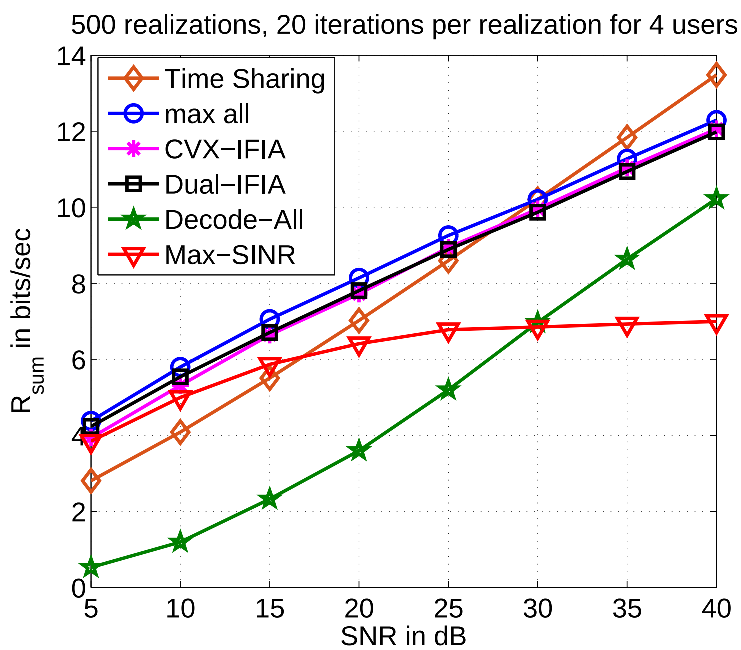

Figure 7·3: Performance of iterative algorithms for IFIA in infeasible scenario: $K = 4$, $N_{\mathrm{Tx}}^{[k]} = N_{\mathrm{Rx}}^{[k]} = 2$, $M^{[k]} = 2, \forall k$.

Chapter 8

Conclusions

8.1 Summary of the thesis

In this thesis, we established an uplink-downlink duality and a source-channel duality under the framework of IF. For uplink-downlink duality, we showed IF can achieve the same sum rate in Gaussian MIMO MAC and its dual Gaussian MIMO BC. For source-channel duality, we showed IF can achieve the same rate tuples in Gaussian MIMO MAC and its dual Gaussian distributed source coding problem. Both duality relationships allow us to better understand the roles of IF architecture in the downlink channel coding problem and distributed Gaussian source coding problem. In the downlink channel, we extended the existing IF beamforming results to allow asymmetric power and asymmetric rates. We designed a DPC technique for IF in the downlink channel based on a known SIC technique from the uplink channel. We showed a constant gap result for the IF downlink computation sum rate and proposed an iterative optimization algorithm based on the uplink-downlink duality result. Our simulation results showed that IF in the downlink channel not only provides an advantage in transmission rate but also has an advantage in lowering the maximum power across antennas. For IF source-channel duality, we extended the existing IF source coding results to allow asymmetric distortions, asymmetric rates, and successive cancellation. We proposed a source-channel duality relationship between successive IF for distributed Gaussian source coding and successive IF for the Gaussian MIMO MAC. We also showed that, even without successive cancellation, the rates of IF source coding and IF channel still lie within a constant gap of one another.

Using the dualities as connections, we can generalize existing results from a well-understood model, e.g. IF in MIMO MAC, to help us better understand and design IF in other models. We can use the basic IF models studied in this thesis, like the IF MAC and IF BC, as building block to further develop IF for more complicated networks.

For IFIA in Gaussian interference channel, we developed linear alignment strategies using a recently developed analytical approach for integer-forcing interference alignment. We explored both signal space alignment and signal scale alignment. Numerical results demonstrate the advantages for integer-forcing interference alignment in both feasible scenario and infeasible scenario compare to existing signal space alignment algorithms such as Max-SINR algorithm. The algorithms further provide insight of interference alignment with static channel realization and low/moderate SNR regime.

8.2 Future Directions

For IF dualities established in this thesis, both uplink-downlink duality and a source-channel duality are *formula duality*. The connections are built upon the rate expressions only. One interesting direction is to further establish a stronger *functional duality* where the duality connections are established on the codebook. Another interesting direction is to establish a *duality loop* by filling the missing bottom left corner in Figure 1·1. However, the *broadcast source coding problem* for IF is not well defined yet. Potential future works can involve finding the correct system model and developing duality connection for IF in the BC source coding problem.

For IFIA, there are many interesting open problems. First, theoretical analysis for the performance of IFIA remains to be explored. The extension of IFIA among multiple channel realizations is also missing. It is also an interesting problem to explore the performance of IFIA in large interference networks.

Appendix A

Aligned LLL lattice reduction algorithms for general $M^{[k]}$

This part we will introduce two additional aligned LLL lattice reduction algorithms for general case where $M^{[k]} \geq 2$ (recall $M^{[k]}$ is the number of combinations decoded at each receiver). In Section 7.5, we introduced Aligned LLL Method-I which works only for $M^{[k]} = 2$. Here, we will introduce Aligned LLL Method-II and Method-III which work for $M^{[k]} \geq 2$. We then compare their performance in terms of sum rate for $M^{[k]} = 2$.

Recall that the effective noise power is given by (7.12) as

$$\left(\sigma_{\text{eff},i}^{[k]}\right)^2 \triangleq \left\|\mathbf{F}^{[k]}\mathbf{a}_i^{[k]}\right\|^2 \tag{A.1}$$

where $\mathbf{F}^{[k]} = \left(\mathbf{P}^{-1} + \mathbf{V}^{\mathsf{T}}\mathbf{H}^{[k]\mathsf{T}}\mathbf{H}^{[k]}\mathbf{V}\right)^{-\frac{1}{2}}$. We can expand (A.1) as

$$\left(\sigma_{\text{eff},i}^{[k]}\right)^2 = \left\|a_{i,k}^{[k]}\mathbf{f}_k^{[k]} + \mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{i,\sim k}^{[k]}\right\|^2 \tag{A.2}$$

where $\mathbf{a}_{i,\sim k}^{[k]}$ is a subvector of $\mathbf{a}_i^{[k]}$ without the k^{th} element, $\mathbf{f}_k^{[k]}$ is the k^{th} column of $\mathbf{F}^{[k]}$, and $\mathbf{F}_{\sim k}^{[k]}$ is a submatrix of $\mathbf{F}^{[k]}$ with the k^{th} column removed. The algorithm for the general case is summarized in Algorithm 5. To develop intuition, we now describe our algorithm for the special case of $M^{[k]} = 2$ and focus on operations at the first receiver (i.e., decoding $\mathbf{x}^{[1]}$). This is equivalent to imposing the following structure on matrix $\mathbf{A}^{[1]}$

$$\mathbf{A}^{[1]} = \begin{bmatrix} a_{1,1}^{[1]} & c_1\, \mathbf{a}_{1,\sim 1}^{[1]\dagger} \\ a_{2,1}^{[1]} & c_2\, \mathbf{a}_{1,\sim 1}^{[1]\dagger} \end{bmatrix} \Leftrightarrow \mathbf{A}_{\sim 1}^{[1]} = \begin{bmatrix} c_1\, \mathbf{a}_{1,\sim 1}^{[1]\dagger} \\ c_2\, \mathbf{a}_{1,\sim 1}^{[1]\dagger} \end{bmatrix} \tag{A.3}$$

where $c_1, c_2 \in \mathbb{Z}$, $c_2 a_{1,1}^{[1]} \neq c_1 a_{2,1}^{[1]}$ and $\mathbf{a}_{1,\sim 1}^{[1]} = \begin{bmatrix} a_{1,2}^{[1]} & \cdots & a_{1,K}^{[1]} \end{bmatrix}$ contains the integer coefficients of the interfering codewords corresponding to columns $\mathbf{f}_2^{[1]}, \ldots, \mathbf{f}_K^{[1]}$. We now present two lattice reduction methods.

Aligned LLL Method-II

In this method, the receiver finds a near-optimal integer vector $\mathbf{a}_1^{[1]}$ by attempting to minimize $\left(\sigma_{\text{eff},1}^{[1]}\right)^2$ using the LLL algorithm [Lenstra et al., 1982]. It then extracts the integer coefficients $\mathbf{a}_{1,\sim 1}^{[1]}$ for the interfering codewords from $\mathbf{a}_1^{[1]}$ and uses these to find $\mathbf{a}_2^{[1]}$ by minimizing $\left(\sigma_{\text{eff},2}^{[1]}\right)^2$ using the LLL algorithm while satisfying the decodability constraint in (A.3).

In other words, the receiver first finds $\mathbf{a}_{1,\sim 1}^{[1]*}$ using

$$\mathbf{a}_1^{[1]*} = \arg \min_{\mathbf{a}_1^{[1]} \in \mathbb{Z}^K} \left\| \mathbf{F}^{[1]} \mathbf{a}_1^{[1]} \right\|^2 \tag{A.4}$$

Finally, the receiver finds the best two independent integer-linear combinations of vectors $\mathbf{f}_1^{[1]}$ and $\mathbf{F}_{\sim 1}^{[1]}$ using

$$\begin{aligned} \mathbf{b}_1^* &= \arg \min_{\mathbf{b} \in \mathbb{Z}^2} \left\| \tilde{\mathbf{F}}^{[1]} \mathbf{b} \right\|^2, \\ \mathbf{b}_2^* &= \arg \min_{\substack{\mathbf{b} \in \mathbb{Z}^2 \\ \text{rank}[\mathbf{b}_1^* \ \ \mathbf{b}] = 2}} \left\| \tilde{\mathbf{F}}^{[1]} \mathbf{b} \right\|^2 \end{aligned} \tag{A.5}$$

where $\tilde{\mathbf{F}}^{[1]} = [\mathbf{f}_1^{[1]} \ \mathbf{F}_{\sim 1}^{[1]} \mathbf{a}_{1,\sim 1}^{[1]*}]$ where $\mathbf{a}_{1,\sim 1}^{[1]*}$ is the subvector of $\mathbf{a}_1^{[1]*}$ with the first element removed. Finally, the integer matrix $\mathbf{A}_1^{[1]}$ is given by

$$\mathbf{A}_1^{[1]} = \begin{bmatrix} b_{1,1}^* & b_{1,2}^* \mathbf{a}_{1,\sim 1}^{[1]*} \\ b_{2,1}^* & b_{2,2}^* \mathbf{a}_{1,\sim 1}^{[1]*} \end{bmatrix} \tag{A.6}$$

Aligned LLL Method-III

The only difference with Method-II is that, instead of finding the best integer vector $\mathbf{a}_1^{[1]*}$ that minimizes $\left(\sigma_{\text{eff},i}^{[1]}\right)^2$ then extracting the coefficients of interference codewords (i.e. $\mathbf{a}_{1,\sim 1}^{[1]*}$), the receiver first aligns the interference codewords to minimize $\sigma_{\text{eff},i}^{[1]2}$ by choosing

$$\mathbf{a}_{1,\sim 1}^{[1]*} = \arg\min_{\mathbf{a}_{1,\sim 1}^{[1]}} \|\mathbf{F}_{\sim 1}^{[1]} \mathbf{a}_{1,\sim 1}^{[1]}\|^2 \tag{A.7}$$

then computes $\mathbf{b}_1^*, \mathbf{b}_2^*$ and $\mathbf{A}_1^{[1]}$ from (A.5) and (A.6), respectively. It is worth noting that both methods run in polynomial time as they only utilize two calls of the LLL algorithm.

Comparison between different aligned LLL methods

A comparison between the performance of three methods are shown in Table A.1 for $\mathsf{SNR} = 25$ dB and $M^{[k]} = 2$. From Table A.1, none of the methods are consistently better (as the maximum is strictly higher than any one of them).

	Method-I	Method-II	Method-III	Best
3 users	9.8316	9.757 (-0.758%)	9.6541(-1.805%)	10.0808(+2.53%)
4 users	8.9254	8.5276(-4.45%)	8.1259(-8.95%)	9.3343(+4.58%)

Table A.1: The sum rate (in bits/Sec/Hz) for different aligned LLL lattice reduction methods at 25 dB for IFIA

Algorithm 5 Generalized Aligned LLL for $M^{[k]}$ integer-combinations (Method-II&III)

1. Step1: Using the LLL algorithm, find the shortest $M^{[k]}-1$ vectors in the lattice $\mathbf{F}^{[k]}$
   Method-II:
   $$i)\ \mathbf{a}_i^{[k]*} = \arg\min_{\mathbf{a}_i^{[k]}} \|\mathbf{F}^{[k]}\mathbf{a}_i^{[k]}\|^2,\ i = 1,\ldots,M^{[k]}-1$$
   $$ii)\ \mathbf{a}_{i,\sim k}^{[k]*} = [a_{i,\sim k}^{[k]*},\ldots,a_{i,K}^{[k]*}],\qquad i = 1,\ldots,M^{[k]}-1$$
   Method-III: $\quad \mathbf{a}_1^{[1]*} = \arg\min_{\mathbf{a}_{i,\sim k}^{[k]}} \|\mathbf{F}_{\sim k}^{[k]}\mathbf{a}_{i,\sim k}^{[k]}\|^2$
2. Step2: Using the LLL algorithm, find the $M^{[k]}$ shortest vectors in the lattice $\tilde{\mathbf{F}}^{[k]} = [\mathbf{f}_k^{[k]}\ \mathbf{F}_{\sim k}^{[k]}\bar{\mathbf{A}}^{[k]}]$
   $$\mathbf{b}_i^* = \arg\min_{\mathbf{b}_i:\mathrm{rank}[\mathbf{b}_1^*,\ldots,\mathbf{b}_{i-1}^*]=i} \|\tilde{\mathbf{F}}^{[k]}\mathbf{b}_i\|^2,\ i = 1,\ldots,M^{[k]}$$
   where $\bar{\mathbf{A}}^{[k]} = [\mathbf{a}_{1,\sim k}^{[k]*},\ldots,\mathbf{a}_{M^{[k]}-1,\sim k}^{[k]*}]$
3. Step3: Calculate the integer matrix $\mathbf{A}^{[k]}$ using
   $$\tilde{\mathbf{A}}^{[k]} = \begin{bmatrix} 1 & \mathbf{0}_{M^{[k]}-1}^{\dagger} \\ \mathbf{0}_{K-1} & \bar{\mathbf{A}}_{\sim k}^{[k]} \end{bmatrix} [\mathbf{b}_1^* \quad ,\ldots, \quad \mathbf{b}_{M^{[k]}}^*]$$
   $$\mathbf{A}^{[k]} = \pi_{\ell,k}(\tilde{\mathbf{A}}^{[k]})$$
   where $\pi_{\ell,k}(\tilde{\mathbf{A}}^{[k]})$= exchanges columns ℓ and k of $\tilde{\mathbf{A}}^{[k]}$

Appendix B

Proof of Lemma 6

Assuming $\mathbf{H}_\mathrm{u}$, $\mathbf{A}_\mathrm{u}$ and $\mathbf{P}_\mathrm{u}$ are given in the uplink channel, the optimal equalization matrix $\mathbf{B}_\mathrm{u}$ is a quadratic problem with a closed-form solution

$$\mathbf{B}_{\mathrm{u,opt}} = \mathbf{A}_\mathrm{u}\mathbf{P}_\mathrm{u}^\top\mathbf{H}_\mathrm{u}^\top(\mathbf{I}+\mathbf{H}_\mathrm{u}\mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\top)^{-1}. \tag{B.1}$$

Recall that the effective noise power is given as:

$$\sigma^2_{\mathrm{u,eff},j} \triangleq \|\mathbf{b}_{\mathrm{u},j}^\top\|^2 + \left(\mathbf{b}_{\mathrm{u},j}^\top\mathbf{H}_\mathrm{u} - \mathbf{a}_{\mathrm{u},j}^\top\right)\mathbf{P}_\mathrm{u}\left(\mathbf{b}_{\mathrm{u},j}^\top\mathbf{H}_\mathrm{u} - \mathbf{a}_{\mathrm{u},j}^\top\right)^\top \tag{B.2}$$

where $\mathbf{b}_{\mathrm{u},j}$ and $\mathbf{a}_{\mathrm{u},j}$ are the j^{th} row vector of $\mathbf{B}_\mathrm{u}$ and $\mathbf{A}_\mathrm{u}$, separately. Put (B.1) into (B.2), we can rewrite (B.2) as

$$\sigma^2_{\mathrm{u,eff},j} \triangleq \mathbf{a}_{\mathrm{u},j}^\top\bigg(\mathbf{P}_\mathrm{u} - \mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\top(\mathbf{I}+\mathbf{H}_\mathrm{u}\mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\top)^{-1}\mathbf{H}_\mathrm{u}\mathbf{P}_\mathrm{u}\bigg)\mathbf{a}_{\mathrm{u},j} \tag{B.3}$$

Let $\mathbf{K} = (\mathbf{P}_\mathrm{u} - \mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\top(\mathbf{I}+\mathbf{H}_\mathrm{u}\mathbf{P}_\mathrm{u}\mathbf{H}_\mathrm{u}^\top)^{-1}\mathbf{H}_\mathrm{u}\mathbf{P}_\mathrm{u})^{\frac{1}{2}}$. Let $\lambda_\ell(\mathbf{K})$ represents the ℓ^{th} successive minimum of $\mathbf{K}$. From [Ordentlich et al., 2012][Theorem 4], it is known that

$$\prod_{\ell=1}^{L}\lambda_\ell(\mathbf{K})^2 \le L^L\|\det(\mathbf{K})\|^2 \tag{B.4}$$

By choosing $\mathbf{a}_{\mathrm{u},1},\cdots,\mathbf{a}_{\mathrm{u},L}$ optimally, we have $\|\mathbf{K}\mathbf{a}_{\mathrm{u},\ell}\| = \lambda_\ell(\mathbf{K})$.

$$\sum_{\ell=1}^{L} R_{\mathrm{u},\ell} = \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \sum_{\ell=1}^{L}\frac{1}{2}\log_2(\sigma^2_{\mathrm{u,eff},j}) \tag{B.5}$$

$$= \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \frac{1}{2}\log_2(\prod_{\ell=1}^{L}\sigma^2_{\mathrm{u,eff},\ell}) \tag{B.6}$$

$$= \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \frac{1}{2}\log_2(\prod_{\ell=1}^{L}\|\mathbf{K}\mathbf{a}_{\mathrm{u},\ell}\|^2) \tag{B.7}$$

$$= \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \frac{1}{2}\log_2(\prod_{\ell=1}^{L}\lambda_\ell(\mathbf{K})^2) \tag{B.8}$$

$$\geq \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \frac{1}{2}\log_2(L^L\|\det(\mathbf{K})\|^2) \tag{B.9}$$

$$= \frac{1}{2}\log_2(\det(\mathbf{P}_{\mathrm{u}})) - \frac{1}{2}\log_2(\|\det(\mathbf{K})\|^2) - \frac{L}{2}\log_2(L) \tag{B.10}$$

$$= \frac{1}{2}\log_2\left(\det\left(\mathbf{P}_{\mathrm{u}}(\mathbf{P}_{\mathrm{u}} - \mathbf{P}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}}(\mathbf{I} + \mathbf{H}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}})^{-1}\mathbf{H}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}})^{-1}\right)\right) - \frac{L}{2}\log_2(L) \tag{B.11}$$

$$= \frac{1}{2}\log_2\left(\det\left(\mathbf{I} - \mathbf{P}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}}(\mathbf{I} + \mathbf{H}_{\mathrm{u}}\mathbf{P}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}})^{-1}\mathbf{H}_{\mathrm{u}}\right)^{-1}\right) - \frac{L}{2}\log_2(L) \tag{B.12}$$

$$= \frac{1}{2}\log_2\det\left(\mathbf{I} + \mathbf{P}_{\mathrm{u}}\mathbf{H}_{\mathrm{u}}^{\mathsf{T}}\mathbf{H}_{\mathrm{u}}\right) - \frac{L}{2}\log_2(L) \tag{B.13}$$

where the last step proof uses the Woodbury matrix identity.

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EDUCATION

- *Boston University, College of Engineering*, Boston, MA
  - Ph.D. in Electrical Engineering Sept. 2011 - Sept. 2016 (expected)
  - M.S. in Electrical Engineering Sept. 2011 - Dec. 2014
- *Polytechnic Institute of New York University*, New York City, NY
  - B.S. in Electrical Engineering Jan. 2008 - May 2011

PROFESSIONAL EXPERIENCE

Boston University, Department of Electrical and Computer Engineering
Graduate Research Assistant Jan. 2012 - Jan. 2016

- OFDM-based wireless open access research platform implementation
  - A complete wireless transmission system based on orthogonal frequency-division multiplexing (OFDM) is developed. The performance of the system is tested using wireless open access research platform (WARP) for real-time transmission.
  - Implemented compute-and-forward scheme as an interference management method.
  - Designed algorithms and signal processing methods for channel estimation, payload detection, carrier frequency offset management, phase offset correction and maximum-likelihood decoding under the framework of compute-and-forward.
- Collision scheduling for cellular networks
  - Designed collision scheduling and interference alignment algorithms to solve collision issues in wireless communication.

  - Modeled collision scheduling problem in cellular network using graph theory.
  - Proposed an integer programming formulation for the scheduling problem as well as a dynamic programming algorithm that can solve it in pseudo-polynomial time.
- Designed algorithms for interference alignment using lattice reduction
  - Developed aligned lattice reduction methods for interference alignment in wireless interference channel.
  - Developed iterative optimization algorithms for channel beamforming and equalization.
- Established uplink-downlink duality and source-channel duality for integer-forcing coding scheme
  - Established an uplink-downlink duality for integer-forcing coding scheme between uplink multiple-access channel and downlink broadcast channel.
  - Established a source-channel duality for integer-forcing coding scheme between uplink multiple-access channel coding problem and Gaussian distributed source coding problem.
  - Developed iterative optimization algorithms to improve wireless transmission in both the uplink channel and downlink channel.

Polytechnic Institute of New York University, Electrical Engineering Department
Undergraduate Research Assistant May 2010 - Jan. 2011

- Software-defined Radio design for Cooperative Communications
  - Simulated multiple interference levels and their influence on multiple-access communication using MATLAB and universal software radio peripheral (USRP2).

RESEARCH INTERESTS

- Information Theory.
- Algorithm design for physical layer channel coding and source coding.
- Interference management for wireless communication.

PUBLICATIONS

1. Wenbo He and Bobak Nazer. Integer-Forcing Source Coding: Successive Cancellation and Source-Channel Duality *Proceedings of the IEEE International Symposium on Information Theory (ISIT2016)*, Barcelona, Spain, July 2016.

2. Wenbo He, Bobak Nazer and Shlomo Shamai. Dirty-paper integer-forcing *The 53rd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, Monticello, IL, October 2015.

3. Wenbo He, Chen Feng and Bobak Nazer. Collision Scheduling for Cellular Networks with Spatial Connectivity Constraints *The 82nd IEEE Vehicular Technology Conference (VTC 2015-Fall)*, Boston, MA, September 2015.

4. Wenbo He, Islam El Bakoury and Bobak Nazer. Integer-forcing interference alignment: Iterative optimization via aligned lattice reduction *IEEE International Symposium on Information Theory (ISIT 2015)*, Hong Kong, China, June 2015.

5. Wenbo He, Chen Feng, Corina I Ionita and Bobak Nazer. Collision scheduling for cellular networks *IEEE International Symposium on Information Theory (ISIT 2015)*, Hong Kong, China, June 2015.

6. Wenbo He, Bobak Nazer and Shlomo Shamai. Uplink-downlink duality for integer-forcing: Effective SINRs and iterative optimization *The 15th IEEE International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, Toronto, Canada, June 2014.

7. Wenbo He, Bobak Nazer and Shlomo Shamai. Uplink-downlink duality for integer-forcing *IEEE International Symposium on Information Theory (ISIT 2014)*, Honolulu, HI, July 2014.

8. Wenbo He. Abstract: Software-defined Radio for Cooperative Communications *NYU-POLY 2010 Undergraduate Abstracts*, 2010.